PHILOSOPHICAL

TRANSACTIONS

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PART I.

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MDCCCXLI.

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THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

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upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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ROYAL MEDALS.

HER MAJESTY QUEEN VICTORIA, in restoring the Foundation of the Royal Medals, has been graciously pleased to approve of the following regulations for the award of them:

That the Royal Medals be given for such papers only as have been presented to the Royal Society, and inserted in their Transactions.

That the triennial Cycle of subjects be the same as that hitherto in operation: viz.

1. Astronomy; Physiology, including the Natural History of Organized Beings.
2. Physics; Geology or Mineralogy.
3. Mathematics; Chemistry.

That, in case no paper, coming within these stipulations, should be considered deserving of the Royal Medal, in any given year, the Council have the power of awarding such Medal to the author of any other paper on either of the several subjects forming the Cycle, that may have been presented to the Society and inserted in their Transactions; preference being given to the subjects of the year immediately preceding: the award being, in such case, subject to the approbation of Her Majesty.

The Council propose to give one of the Royal Medals in the year 1841 for the most important unpublished paper in Mathematics, communicated to the Royal Society for insertion in their Transactions after the termination of the Session in June 1838, and prior to the termination of the Session in June 1841.

The Council propose also to give one of the Royal Medals in the year 1841 for the most important unpublished paper in Chemistry, communicated to the Royal Society

for insertion in their Transactions after the termination of the Session in June 1838, and prior to the termination of the Session in June 1841.

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The Council propose also to give one of the Royal Medals in the year 1844 for the most important unpublished paper in Chemistry, communicated to the Royal Society for insertion in their Transactions after the termination of the Session in June 1841, and prior to the termination of the Session in June 1844.

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PHILOSOPHICAL TRANSACTIONS.

I. *Supplement to a Paper "On the Theoretical Explanation of an apparent new Polarity in Light."* By G. B. Airy, *Esq. M.A. F.R.S., Astronomer Royal.*

Received October 24,—Read November 19, 1840.

In the Second Part of the Transactions of the year 1840, the Royal Society has published a memoir by me, explaining, on the undulatory theory of light, the apparent new polarity observed by Sir David Brewster; which explanation is based upon the assumption that the spectrum is viewed out of focus; an assumption which corresponded to the circumstances of my own observations, and to those of some other persons. Since the publication of that memoir, I have been assured by Sir David Brewster that the phenomenon was most certainly observed with great distinctness when the spectrum was viewed so accurately in focus that many of Fraunhofer's finer lines could be seen. This observation appeared to be contradictory to those of Mr. Talbot, cited by me in page 226 of the memoir, as well as to my own. With the view of removing the obscurity that still appeared to embarrass this subject, I have continued the theoretical investigation for that case which was omitted in the former memoir, namely, when the spectrum is viewed in focus, or when $a = 0$ (page 229); and I have arrived at a result which appears completely to reconcile the seemingly conflicting statements.

In the following investigation I shall use the symbols and the formulæ of the former memoir (as far as they apply) without further reference.

The value of ϱ in page 228 becomes, on making $a = 0$,

$$\varrho = e - \frac{b}{e} y,$$

and the disturbance of ether, on the point of the retina whose distance from the geometrical image is b, produced by a small portion δy of the front of the wave, is

$$\delta y \times \sin \frac{2\pi}{\lambda} (v t - \varrho)$$

or

$$\delta y \times \sin \frac{2\pi}{\lambda} \left(v t - e + \frac{b}{e} y \right),$$

and therefore the whole disturbance of ether on the point of the retina, produced by that part of the pupil which is not covered by any retarding plate, is

$$\int_y \sin\frac{2\pi}{\lambda}\left(vt - e + \frac{b}{e}y\right),$$

the limits of the integral being the values of y corresponding to the boundaries of the part of the pupil not covered by a retarding plate.

But if a portion of the pupil be covered by a plate producing the retardation R (expressed as an angle) in the phase of the wave, the expression to be integrated through the limits proper for the covered part will be

$$\int_y \sin\left(\frac{2\pi}{\lambda}(vt - e) - \mathrm{R}\right)$$

or

$$\int_y \sin\left(\frac{2\pi}{\lambda}\left(vt - e + \frac{b}{e}y\right) - \mathrm{R}\right).$$

Let the limits of the pupil be from $-h$ to $+h$, without regard to the other ordinate upon its surface (which amounts to supposing the form of the pupil to be a parallelogram), and let the part which depends on R be taken between the limits 0 and $+h$ (which amounts to supposing that half of the pupil to be covered which is on the side on which b is considered positive). Then the whole disturbance of the ether is

$$\begin{aligned}
&\int_y \sin\frac{2\pi}{\lambda}\left(vt - e + \frac{b}{e}y\right) \text{ from } y = -h \text{ to } y = 0\\
&+\int_y \sin\left(\frac{2\pi}{\lambda}\left(vt - e + \frac{b}{e}y\right) - \mathrm{R}\right) \text{ from } y = 0 \text{ to } y = +h\\
&= \frac{\lambda e}{2\pi b}\left\{\cos\frac{2\pi}{\lambda}\left(vt - e - \frac{bh}{e}\right) - \cos\frac{2\pi}{\lambda}(vt - e) + \cos\left(\frac{2\pi}{\lambda}(vt - e) - \mathrm{R}\right)\right.\\
&\qquad \left. - \cos\left(\frac{2\pi}{\lambda}\left(vt - e + \frac{bh}{e}\right) - \mathrm{R}\right)\right\}.
\end{aligned}$$

The coefficient of $\cos\frac{2\pi}{\lambda}(vt - e)$ is

$$\begin{aligned}
&\frac{\lambda e}{2\pi b}\left\{\cos\frac{2\pi}{\lambda}\cdot\frac{bh}{e} - 1 + \cos\mathrm{R} - \cos\left(\frac{2\pi}{\lambda}\cdot\frac{bh}{e} - \mathrm{R}\right)\right\}\\
&= -\frac{\lambda e}{2\pi b}\left\{1 - \cos\frac{2\pi}{\lambda}\cdot\frac{bh}{e} - \cos\mathrm{R} + \cos\frac{2\pi}{\lambda}\cdot\frac{bh}{e}\times\cos\mathrm{R} + \sin\frac{2\pi}{\lambda}\cdot\frac{bh}{e}\times\sin\mathrm{R}\right\}\\
&= -\frac{\lambda e}{2\pi b}\left\{\left(1 - \cos\frac{2\pi}{\lambda}\cdot\frac{bh}{e}\right)\times(1 - \cos\mathrm{R}) + \sin\frac{2\pi}{\lambda}\cdot\frac{bh}{e}\times\sin\mathrm{R}\right\}\\
&= -\frac{\lambda e}{2\pi b}4\sin\frac{\pi}{\lambda}\cdot\frac{bh}{e}\times\sin\frac{\mathrm{R}}{2}\times\left\{\sin\frac{\pi}{\lambda}\cdot\frac{bh}{e}\times\sin\frac{\mathrm{R}}{2} + \cos\frac{\pi}{\lambda}\cdot\frac{bh}{e}\times\cos\frac{\mathrm{R}}{2}\right\}\\
&= -\frac{2\lambda e}{\pi b}\cdot\sin\frac{\pi bh}{\lambda e}\cdot\sin\frac{\mathrm{R}}{2}\cdot\cos\left(\frac{\pi bh}{\lambda e} - \frac{\mathrm{R}}{2}\right);
\end{aligned}$$

and the coefficient of $\sin\frac{2\pi}{\lambda}(vt-e)$ is

$$\frac{\lambda e}{2\pi b}\left\{\sin\frac{2\pi}{\lambda}\cdot\frac{bh}{e}+\sin R+\sin\left(\frac{2\pi}{\lambda}\cdot\frac{bh}{e}-R\right)\right\}$$

$$=\frac{\lambda e}{2\pi b}\left\{\sin\frac{2\pi}{\lambda}\cdot\frac{bh}{e}\times(1+\cos R)+\sin R\times\left(1-\cos\frac{2\pi}{\lambda}\cdot\frac{bh}{e}\right)\right\}$$

$$=\frac{2\lambda e}{\pi b}\cdot\sin\frac{\pi bh}{\lambda e}\cdot\cos\frac{R}{2}\cdot\left\{\cos\frac{\pi bh}{\lambda e}\cdot\cos\frac{R}{2}+\sin\frac{R}{2}\cdot\sin\frac{\pi bh}{\lambda e}\right\}$$

$$=\frac{2\lambda e}{\pi b}\cdot\sin\frac{\pi bh}{\lambda e}\cdot\cos\frac{R}{2}\cdot\cos\left(\frac{\pi bh}{\lambda e}-\frac{R}{2}\right).$$

And the intensity of light on the point of the retina, which is represented by the sum of the squares of these coefficients, is

$$\frac{4\lambda^2e^2}{\pi^2b^2}\cdot\sin^2\frac{\pi bh}{\lambda e}\cdot\cos^2\left(\frac{\pi bh}{\lambda e}-\frac{R}{2}\right).$$

For convenience, put $\frac{\pi bh}{\lambda e}=w$, and omit the constant factor $4h^2$; the expression becomes then

$$\left(\frac{\sin w}{w}\right)^2\cdot\cos^2\left(w-\frac{R}{2}\right),$$

where it must be borne in mind that w is a multiple of the distance, of the point of the retina at which the intensity is sought, from the geometrical image of the point of light. It must also be borne in mind that this expression gives the intensity on that point of the retina produced by a single point of light, or a single line of light parallel to the bounding edge of the retarding plate.

The following Table contains the values of $\left(\frac{\sin w}{w}\right)^2\cos^2\left(w-\frac{R}{2}\right)$ for every 10° of w, and for every 60° of R. In computing them, w has been expressed in degrees: and the last figure of the numbers contained in the Table is the eighth decimal place.

Table of $\left(\frac{\sin w}{w}\right)^2 \cdot \cos^2\left(w - \frac{R}{2}\right)$.

Values of w.	Values of R.					
	0°	60°	120°	180°	240°	300°
− 175	25	20	8	0	4	17
− 165	229	230	123	17	17	123
− 155	610	738	499	133	6	245
− 145	1050	1553	1286	515	12	280
− 135	1372	2560	2560	1372	184	184
− 125	1413	3527	4261	2882	767	33
− 115	1109	4168	6164	5101	2043	47
− 105	567	4231	7896	7896	4231	567
− 95	84	3618	9032	10912	7378	1964
− 85	104	2453	9216	13631	11282	4519
− 75	1111	1111	8294	15475	15475	8294
− 65	3473	148	6396	15970	19294	13046
− 55	7298	169	3962	14884	22013	18221
− 45	12346	1654	1654	12346	23038	23038
− 35	18020	4796	204	8835	22060	26652
− 25	23474	9402	217	5104	19175	28360
− 15	27778	14887	1994	1994	14886	27778
− 5	30154	20389	5427	231	9996	24959
+ 5	30154	24959	9996	231	5427	20388
+ 15	27778	27778	14887	1994	1994	14887
+ 25	23474	28360	19175	5104	217	9402
+ 35	18020	26651	22060	8835	204	4794
+ 45	12346	23038	23038	12346	1654	1654
+ 55	7298	18220	22013	14884	3962	169
+ 65	3473	13045	19294	15970	6396	148
+ 75	1111	8294	15475	15475	8294	1111
+ 85	104	4519	11282	13631	9216	2453
+ 95	84	1964	7378	10912	9032	3618
+ 105	567	567	4231	7896	7896	4231
+ 115	1109	47	2043	5101	6164	4168
+ 125	1413	33	767	2882	4261	3527
+ 135	1372	184	184	1372	2560	2560
+ 145	1050	279	12	515	1286	1553
+ 155	610	245	6	133	499	738
+ 165	229	123	16	17	123	230
+ 175	25	17	4	0	8	20

It has been deemed unnecessary to continue the Table beyond the values of $w - 175°$ and $+ 175°$, because the values of $\left(\frac{\sin w}{w}\right)^2$ become very small. The greatest maximum of this quantity occurs when $w = 0$; its value (expressing w in terms of the radius) is then $= 1$; the second maximum occurs when $w = \frac{3\pi}{2}$ nearly; its value is then $\left(\frac{2}{3\pi}\right)^2$ nearly $= \frac{1}{22}$ nearly; the amount of which will probably produce an inconsiderable influence on the expressions which we are now about to consider.

The curves in the annexed figure represent, by their ordinates, the values of $\left(\frac{\sin w}{w}\right)^2 \cdot \cos^2\left(w - \frac{R}{2}\right)$; the values of R being continued as far as 720°, in order to exhibit more distinctly to the eye the successive displacements of the principal bows

of the curves. The ordinates, therefore, represent the intensity of light on different points of that small diffused image on the retina which is formed by the light coming from a single point, even when it is seen accurately in focus; the extreme breadth of the image represented in the figure corresponds to 360° of w, or is $= \frac{2\lambda e}{h}$.

If we express the area of each of the curves by summing the ordinates and dividing the sum by thirty-six, we find the following values:

$$\begin{aligned} R = \quad & 0, \text{ area is represented by } 7234 \\ & 60, \text{ area is represented by } 7055 \\ & 120, \text{ area is represented by } 6696 \\ & 180, \text{ area is represented by } 6517 \\ & 240, \text{ area is represented by } 6696 \\ & 300, \text{ area is represented by } 7055. \end{aligned}$$

I shall proceed now to apply these numbers to the explanation of the phenomena in question.

Light is supposed to be incident on the eye from different points of a spectrum, formed in any way: the characteristic of the spectrum as concerned in the present investigation being, that the order of position of the different colours is the same as the order of the successive values of R.

First. Suppose the value of $\frac{2\lambda e}{h}$ to be small, at least in comparison with the distance between those points of the image of the spectrum in which R has changed by 360°.

1. Let $\frac{2\lambda e}{h}$ be exceedingly small. Since the same form of curve recurs for every change of 360° in R and not oftener, it is evident that the succession of bands (if there are any) in the visible image will depend on the changes of 360° in R. Our supposition, therefore, amounts to this; that the extent of the small diffused image is exceedingly less than the interval between the bands (if there are any). Here it is plain that the formation of the broad bands cannot depend on the inequalities of light in the narrow diffused image, but must depend on the quantity of light in the whole of each narrow diffused image considered as a total light from one point of the spectrum. Now the total light is equal for all points. For, as the intensity of light coming from one luminous point and falling on a point of the retina is represented by $\left(\frac{\sin w}{w}\right)^2 . \cos^2\left(w - \frac{R}{2}\right)$, the whole light coming from that luminous point is $\int_w \left(\frac{\sin w}{w}\right)^2 . \cos^2\left(w - \frac{R}{2}\right)$, the limits of the integral being $\pm \infty$. Now this definite integral is independent of R. For

$$\cos^2\left(w - \frac{R}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos R + \cos R . \cos^2 w + \sin R . \cos w . \sin w,$$

and therefore

$$\left(\frac{\sin w}{w}\right)^2 \cos^2\left(w - \frac{R}{2}\right) = \left(\frac{1}{2} - \frac{1}{2}\cos R\right)\left(\frac{\sin w}{w}\right)^2 + \cos R\left(\frac{\cos w \,.\, \sin w}{w}\right)^2 + \sin R \,.\, \frac{\cos w \,.\, \sin^3 w}{w^3};$$

and the whole intensity of light is represented by

$$\left(\frac{1}{2} - \frac{1}{2}\cos R\right)\int_w \left(\frac{\sin w}{w}\right)^2 + \cos R \int_w \left(\frac{\cos w \,.\, \sin w}{w}\right)^2 + \sin R \int_w \frac{\cos w \,.\, \sin^2 w}{w^2},$$

the limits of integration being $\pm \infty$. The last term, changing sign when w changes sign, evidently makes its definite integral $= 0$: the two former may be put in the shape

$$\left(\frac{1}{2} - \frac{1}{2}\cos R\right)\int_w \left(\frac{\sin w}{w}\right)^2 + \frac{1}{2}\cos R \int_{2w} \left(\frac{\sin 2w}{2w}\right)^2.$$

If $\int_w \left(\frac{\sin w}{w}\right)^2$ from $-\infty$ to $+\infty$ be $= S$, then $\int_{2w} \left(\frac{\sin 2w}{2w}\right)^2$ from $-\infty$ to $+\infty$ is also $= S$, and the expression becomes

$$\left(\frac{1}{2} - \frac{1}{2}\cos R\right) \cdot S + \frac{1}{2}\cos R \,.\, S \quad \text{or } \frac{1}{2} S,$$

which is independent of R. The total light, therefore, is independent of R, or is equal at all points; and therefore no bands are produced.

2. But if $\frac{2\lambda e}{h}$, though small, is not exceedingly small, the principal impression may be made upon the eye by the central patch of light from each source, included between the values $w = -180°$, $w = +180°$; while those parts of the light which extend beyond the central patch may be in fact aggregated with the central patches of light from the sources at a small distance on each side. And if the amounts in the central patches from different sources are unequal, while the whole amounts from the different sources are equal, it is evident that a bright central patch from one source may be combined with bright detached parts from another source, while a fainter central patch from that second source may be combined with an insignificant detached part from the first source, and thus the whole inequality of light may be double the inequality of the central patches. Now the amount of the light in the central patch, as we have found, is greatest, and represented by 7234, when $R = 0$ or $= 2n\pi$, and is least, and is represented by 6517, when $R = \pi$ or $= 2n + 1 \,.\, \pi$. The difference of these is $\frac{1}{10}$th of the whole; and therefore the difference of the whole light on each part of the retina, formed by combining the central patch formed by one source with the detached light formed by another source, will be nearly $\frac{1}{5}$th of the whole. This inequality of light is amply sufficient to form conspicuous bands.

The bars thus formed depend upon nothing but the changes in the value of R: it is wholly indifferent whether R increases or diminishes towards the side on which b

is considered positive; that is, it is indifferent whether the retarding plate is applied on the same side as the red end or the violet end of the spectrum. These appear to be the bars seen by Mr. TALBOT and myself when the spectrum was viewed in focus. They require that $\frac{2\lambda e}{h}$ be not large, that is, that the aperture of the pupil ($2h$), or the aperture of the telescope used be not very small; and that the changes of R be not very rapid; that is, that the plate of mica, &c. be thin. These circumstances held in my own experiment. I may add that the dark bands were not black, but merely dusky; as indicated by the numbers above.

Secondly. Suppose the value of $\frac{\lambda e}{h}$ to be comparable with the distance between those points of the image of the spectrum in which R has changed by 360°; for instance, suppose $\frac{\lambda e}{h}$ to be equal to that distance.

1. Let the red end of the external spectrum be on the same side as the retarding plate, that is, on the side on which b is considered positive. Then on the retina the violet end is on that side; or R increases towards the positive side. Let k be the ordinate measured from a fixed point on the retina to the centre of the diffused image of any colour (k being therefore a function of λ), and l the ordinate measured from the same fixed point to the point at which the intensity is to be ascertained; then $k + b = l$, or $b = l - k$, and the intensity produced by any one kind of light is represented by

$$\frac{\sin^2 \frac{\pi h}{\lambda e}(l-k)}{\left\{\frac{\pi h}{\lambda e}(l-k)\right\}^2} \cos^2\left(\frac{\pi h l}{\lambda e} - \frac{\pi h k}{\lambda e} - \frac{R}{2}\right).$$

The sum of the intensities on one point of the retina produced by all the different kinds of light from the adjacent portions of the spectrum will be found by varying k in this expression, and adding together all the values so produced. Now if R increases when k increases (as occurs when the red end of the external spectrum is on the same side as the retarding plate), the last factor $\cos^2\left(\frac{\pi h l}{\lambda e} - \frac{\pi h k}{\lambda e} - \frac{R}{2}\right)$ will undergo very great changes from the combined changes of $\frac{\pi h k}{\lambda e}$ and $\frac{R}{2}$, whatever be the value of l, and the succession of values which it receives will not differ materially for different values of l; the first factor $\frac{\sin^2 \frac{\pi h}{\lambda e}(l-k)}{\left\{\frac{\pi h}{\lambda e}(l-k)\right\}^2}$ will also undergo great changes, but nearly the same for different values of l; and in consequence the aggregate of all the values for different values of k, exhibiting the total intensity of light upon the point l, will be nearly the same.

This aggregation will be represented graphically by supposing the second curve in

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the diagram to be moved towards the right hand, the third to be moved further to the right, &c., and taking the sum of the ordinates of the various curves which are then placed vertically one below the other; it is clear that the large ordinates of one curve will be added to the small ones of another, so as to produce in every part an approximate mean value. If we perform the same operation numerically, combining the last number of the first column in the Table with the last but three in the second column, the last but six in the third column, and so on, to the twelfth column (observing that the numbers in the columns recur after the sixth, or that they may be supposed to recur before the first), and if we remark that by adding the numbers from twelve columns we do in fact combine the intensities from all the diffused images that are in any degree superposed; and if we then divide by twelve, we find the following numbers to represent the intensities:

6884, 6882, 6881, 6879, 6875, 6872, 6870, 6868, 6867, 6868, &c.,

the greatest number being 6884 and the least 6868. It is plain that no bands will be visible here.

2. Let the violet end of the external spectrum be on the same side as the retarding plate. The same algebraic expression holds as in the other case, but there is this important difference in the interpretation, that R (which increases towards the violet end of the spectrum) is greatest in the spectrum on the retina on that side on which k is negative, or when k increases R diminishes. And if $\frac{\lambda e}{h}$ be equal to the change of k corresponding to a change of 2π in R, or if $\frac{\pi h}{\lambda e} \cdot \frac{\lambda e}{h}$ (or π) be equal to the change of $\frac{\pi h k}{\lambda e}$ corresponding to a change of 2π in R, or of π in $\frac{R}{2}$; then the changes of $\frac{\pi h k}{\lambda e}$ and of $\frac{R}{2}$ exactly destroy each other; $\frac{\pi h k}{\lambda e} + \frac{R}{2} =$ a constant C, and the whole intensity of light on a given point will be found by aggregating all the quantities

$$\frac{\sin^2 \frac{\pi h}{\lambda e}(l-k)}{\left\{\frac{\pi h}{\lambda e}(l-k)\right\}^2} \cdot \cos^2\left(\frac{\pi h l}{\lambda e} - C\right),$$

giving different values to k. As the second factor is independent of k, and as the changes of the first caused by changing the values of k will be similar (to the extent to which the light is sensible), whatever be the value of l, it follows that the aggregate will be expressed by the form $B \cos^2\left(\frac{\pi h l}{\lambda e} - C\right)$. This expression denotes that there will be light of all degrees of intensity from the brightest B to zero or total darkness; and that the whole of the changes will recur (or the dark bands will recur) when $\frac{\pi h l}{\lambda e}$ has changed by 2π, or when l has changed by $\frac{2\lambda e}{h}$.

This combination will be represented graphically by drawing back the second curve

of the diagram by 30°, the third by 60°, and so on; and taking the sum of the ordinates which are then vertically one below the other. It is evident that the ordinates *zero* correspond throughout. If we perform the same operation numerically, combining the first number of the first column in the table with the fourth number of the second column, the seventh number of the third column, &c., and if we then divide the sum by twelve, we find the following numbers:

13646, 12829, 11295, 9227, 6875, 4524, 2456, 921, 105, 105, 921, &c.;

the greatest number being 13646, or a little greater, and the least being 0.

It is evident that these numbers denote the formation of most vivid black and bright bands.

The case which we have taken $\left(\text{when } \frac{\lambda e}{h} \text{ is exactly equal to the change of } k \text{ corresponding to a change of } 2\pi \text{ in R}\right)$ is the most favourable for the production of bands; but it will easily be understood that, in consequence of the small extent of the diffused image, conspicuous bands may be formed when the change of k corresponding to a change of 2π in R is sensibly greater or less than $\frac{\lambda e}{h}$.

The interval between the bands is $\frac{2\lambda e}{h}$, and is, therefore, usually small. They will, however, be made broader by making h small, that is, by contracting the aperture of the pupil, or by using a telescope with a limited object-glass. The value of R changes through 2π with no greater change in the quality of light than that produced by passing from one part of the spectrum to another part distant (on the retina) by $\frac{2\lambda e}{h}$, and therefore the retarding plate must be comparatively thick.

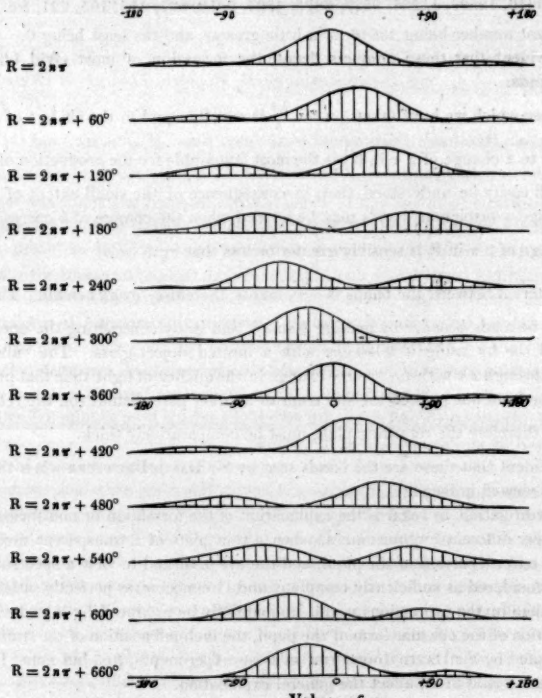
It is evident that these are the bands seen by Sir David Brewster when the spectrum was viewed in focus.

The investigation, as regards the explanation of the formation or non-formation of bands under different circumstances, when a thin plate of a transparent medium is placed to cover a portion of the pupil, and the eye is turned to view a spectrum, may now be considered as sufficiently complete, and (I conceive) as perfectly satisfactory. Some change in the expressions would undoubtedly be produced by introducing the consideration of the circular form of the pupil, the inclined position of the transparent plate adopted by Sir David Brewster in some experiments, &c., but none, I apprehend, which would at all affect the general explanation.

G. B. Airy.

Royal Observatory, Greenwich,
Oct. 23, 1840.

Curves representing by their ordinates the values of $\left(\frac{\sin w}{w}\right)^2 \cos^2\left(w - \frac{R}{2}\right)$.



II. *Contributions to Terrestrial Magnetism.—No.* II.

By Lieut.-Colonel Edward Sabine, *R.A. V.P.R.S.*

Received February 4,—Read February 11, 1841.

§ 3. *Captain* Belcher's *Observations on the west Coast of America, and the adjacent Islands.* § 4. *New Determination of the Magnetic Elements at Otaheite.*

§ 3. *Captain* Belcher's *Observations on the west Coast of America, and the adjacent Islands.*

THE observations, an account of which is now presented to the Society, were made by Captain Edward Belcher, R.N. and the officers of H.M.S. Sulphur, employed in the years 1837 to 1840 in surveying portions of the west coast of North America. The account has been drawn up from the official reports transmitted to the Admiralty, and placed in my hands by the Hydrographer, Captain Beaufort. The services which Captain Belcher and his officers may be expected to render to magnetical science are not terminated, as the Sulphur has not yet returned to England: but the portion now communicated forms a complete series, comprising the results of their labours up to the period of their final departure from the coast of America. The zeal, perseverance, and care with which these have been conducted will be best appreciated by an examination of the details.

Horizontal Intensity.—Captain Belcher joined the Sulphur at Panama in the spring of 1837, receiving from his predecessor, Captain Beechey, a six-inch inclination instrument by Robinson, and several needles for experiments on the horizontal intensity by the method of vibration. He had taken with him from England a nine-inch altitude and azimuth instrument with attached needles, and a five-inch theodolite, both by Cary, which he had employed in former surveys in determining declinations, and had had reason to confide in. Before his departure from Panama on a surveying cruize, which might furnish opportunities of magnetic observation at several stations on the west coast of America between Behring Strait and Peru, the times of vibration of the horizontal needles, eleven in number, were carefully observed, in March 1837, at a convenient spot near the ruins of the Convent of St. Francisco; and these observations were repeated at the same spot on the return of the Sulphur to Panama in October 1838, after an absence of eighteen months. By comparing the times of vibration in March 1837 and October 1838, as given in the subjoined Table, it will be seen that the magnetism of several of the needles had greatly altered in the interim:

Table I.—Comparison of Intensity Needles at Panama in March 1837, and October 1838. Commencing arcs 40°.

Periods.	Designation of the Needles.										
	1.	3.	4.	5.	6.	7.	8.	9.	11.	12.	13.
	s	s	s	s	s	s	s	s	s	s	s
March 1837	625·1	600·9	775·7	472·7	512·6	532·7	470·4	434·3	453·9	373·8	375·2
October 1838....	673·1	608·6	864·0	475·2	514·6	536·8	471·4	439·5	475·8	403·7	395·5

Nos. 5. 6. 7. and 8. had each undergone a small and comparatively insignificant loss of force; but the changes sustained by the other needles, especially by Nos. 1. 4. 11. 12. and 13, were too great to justify the deduction of results, either from a mean of the times of vibration at the two periods, or on the principle of an uniform loss corresponding to equal intervals of time. Unfortunately, Nos. 1. 3. and 4. were amongst those which had been most frequently employed at the stations visited in the cruize; and as an attentive examination of the observations made with them has not furnished, as it sometimes does, the means of discovering when and in what manner the alterations of magnetism took place, I have not attempted to draw from these observations conclusions which could not be otherwise than unsatisfactory. Happily several of the stations were revisited in 1839, when the apparatus was in more perfect order, and the observers having improved by practice, the results are such as leave no other regret for the failure on the first occasion, than what is due to the loss of time and pains. At those stations of the first cruize which were not subsequently visited, we may still derive results from the observations with Nos. 5. 6. 7. and 8, which, though not entitled to equal confidence in respect to precision with the determinations made in the subsequent voyage, are nevertheless well deserving of regard and record. It may be convenient, however, in the relation, to invert the order of succession, and to commence with an account of the second, or principal *magnetic*, voyage.

Having occasion to remain at Panama and its neighbourhood for some months after the needles had been vibrated as above noticed in October 1838, Captain Belcher repeated the observations with the needles specified in the next Table a third time, at the same place as before, on the 16th of March 1839. The times of vibration inserted in this Table were on both occasions in arcs commencing with 40°, which had been the uniform practice with all the needles at the stations visited in the first voyage. Having heard from Captain Beaufort of the attention which Captain Belcher and his officers were giving to magnetic observations, and having been permitted to examine the reports of the observations of the first voyage which had reached the Admiralty on the 1st of January 1839, I wrote to Captain Belcher to recommend that in future he should commence the vibrations at an arc of 20°. This letter was received in Panama early in March, and a double series of observations were made in consequence on the 16th of March, one series commencing with 40° to compare with those of October 1838, and a second commencing with 20°, to correspond with all the ob-

servations which should be subsequently made. The times of vibration commencing with 40° are those inserted in this Table; the series commencing with 20° will be found in its due succession.

Table II.—Comparison of the Intensity Needles at Panama in October 1838, and March 1839*.

Periods.	Designation of the Needles.						
	5.	7.	8.	9.	11.	12.	13.
	s	s	s	s	s	s	s
October 1838....	475·2	536·8	471·4	439·5	475·8	403·7	395·5
March 1839	476·3	536·1	471·4	438·2	474·8	404·6	394·5

This comparison having shown that the seven needles specified in the Table were in a steady magnetic state, Captain Belcher despatched Nos. 7. 8. and 9. to England, to have their times of vibration observed there, and to be returned to him on the coast of California; purposing by this means to attach his series of relative determinations to the great body of results obtained by other observers. The needles were received by me in August 1839, and were vibrated on the 12th and 13th of August at a suitable place near Woolwich, where I also observed the dip at the same time. They reached Captain Belcher again in the following November at Mazatlan. In the meantime the Sulphur had quitted Panama, having on board Nos. 5. 11. 12. and 13, and had visited successively Cocos Island, Oahu one of the Sandwich Islands, Kodiack and Sitka on the north-west coast of America, Fort Vancouver and Baker's Bay in Columbia River, Port Bodega, San Francisco, Monterey, Sta Barbara, San Pedro, San Diego, San Quentin, San Bartholomew, Magdalena Bay, and St. Lucas Bay, arriving at Mazatlan in November. At each of the above-named stations the times of vibration of one or more of the needles were observed, and occasionally of all the four. On the arrival of the three needles which had been sent to England, their times of vibration were observed, in comparison with the others, first at Mazatlan, and a few days afterwards at San Blas, where, more time being available, the comparison was repeated on two different days, viz. on the 6th and 19th of December. From San Blas

* Nos. 1. 3. and 4. are not included in this Table, because the observations on the 16th of March 1839 showed that they were still losing magnetism, and they were not therefore subsequently employed. No. 6. is also omitted, and the cause is explained by a memorandum of Captain Belcher's to the following effect: "No. 6. kept well during the first twenty-one months, and changed suddenly during an excursion to Conchagua in November and December 1838. It was vibrated on the 20th and 22nd of November, and gave consistent results: on the 27th it was carried on horseback up the Amapola hill, 3000 feet above the sea, and on its return on the 22nd of December was found to have lost magnetism equivalent to upwards of 12 seconds in 510 seconds. The surface rock on the Amapola hill was so highly magnetic that no satisfactory observations could be obtained there with the needle." An examination of the subsequent observations of No. 6, compared with those of the other needles, shows that its magnetism was unsteady for many months after this accident, becoming gradually weaker. I have not, therefore, taken into account the observations with this needle, as they do not yield independent results of equal value with the other needles, and there are enough consistent determinations without them.

the Sulphur proceeded to the islands of Socorro and Clarion, to Martins Island one of the Marquesas, and lastly to Bow Island, with which the stations on the west coast of America and its adjacent islands may be considered to have terminated.

To have made this series of magnetic determinations thoroughly complete, the needles should have been taken back to Panama, and their times of vibration should have been re-examined there at the close of the operations; but this proceeding did not consist with other duties. We are, therefore, without that direct evidence of the steady magnetism of the needles, subsequently to the observations at Panama in March 1839, which might have been furnished thereby; but where so many needles are employed, evidence of scarcely inferior weight may be obtained by their intercomparison; especially at stations where the opportunities of observation are favourable, and the probable error of the result with each needle is further diminished by its being derived from repetitions on different days. The observations at Mazatlan and San Blas, on the return of Nos. 7. 8. and 9. from England, furnish one good occasion of this nature; and we may take as a second the observations at Martins Island, being the last station at which they were repeated on different days.

If we divide the squares of the times of vibration of the several needles at Panama by the squares of their times of vibration at Mazatlan, we obtain quotients, which, if the needles were unchanged relatively to each other in the interim, should be identical; or as nearly so as the ordinary errors of observation permit, including therein the diurnal and irregular variations of the magnetic force itself. The times of vibration at San Blas and Panama, similarly treated, supply a similar comparison, in which, however, the quotients will differ in absolute value from the preceding ones, inasmuch as the horizontal intensity of the earth's magnetism is not precisely the same at Mazatlan and San Blas*; but the degree of accordance *with each other* of the second series of quotients will furnish, as in the former case, the required evidence; which is of greater weight in the instance of San Blas than in that of Mazatlan, because the times of vibration at San Blas were derived from observations on two different days, and at Mazatlan from those of a single day only.

Table III.—Intercomparison of the Intensity Needles at Mazatlan and San Blas.

	Quotients.								
	5.	7.	8.	9.	11.	12.	13.	Mean, omitting No. 8.	Weights†.
Panama and Mazatlan	0·930	0·930	0·923	0·924	0·931	0·928	0·925	0·928	2
Panama and San Blas	0·951	.....	0·950	0·956	0·959	0·962	0·956	0·958	3
Difference of each needle from the mean	+ ·002	+ ·002	− ·005	− ·004	+ ·003	·000	− ·003	Panama and Mazatlan.	
	+ ·001	.....	− ·008	− ·002	+ ·001	+ ·004	− ·002	Panama and San Blas.	
Mean difference.....	+ ·001	+ ·002	− ·007	− ·003	+ ·002	+ ·002	− ·002	Allowing the respective weights.	

* The quotients are, in fact, in the two cases, the respective values of the horizontal intensity at Mazatlan and San Blas relatively to the force at Panama taken as unity.

† These are arbitrary weights, assigned according to the number of *days* employed in each comparison.

No. 8. is the only needle which presents a difference from the other needles exceeding in value a five-hundredth part of the time of vibration. If therefore the magnetism of Nos. 5. 7. 9. 11. 12. and 13. suffered any change in the interval comprehended by the comparison, the alteration must have taken place to an equivalent amount in each of the needles: a coincidence the less probable, because they had all been previously exposed to greater extremes of natural temperature than were experienced at the stations visited between Panama and San Blas.

The difference which No. 8. presents from the mean of the other needles is equivalent to 1ˢ·7 in 480 seconds, its time of vibration, or to a proportional loss of magnetism. It does happen occasionally that results with the same needle corrected for temperature, will differ from each other to this amount, in cases when the subsequent return to the original time of vibration manifests that the magnetism of the needle has undergone no change, or at least no permanent change: but as the difference exceeds the probable error of observation, as will be presently shown, and as moreover nearly the same difference appears at all the subsequent stations, when the results with No. 8. are compared with those of the other needles, I have regarded it as an indication of an actual loss of magnetism sustained by No. 8., at some time between the observations at Panama in March 1839, and those at Mazatlan in November of the same year, rendering that needle less fit than the others for intermediate deductions: and I have allowed for this loss at all the stations subsequently to San Blas, by deducting ·00312 from the logarithm of the square of its time of vibration.

The observations at Mazatlan with No. 7, for the comparison of its time of vibration with that of the other needles, were made on the 29th of November. On the 30th of November and 2nd of December, this needle was employed in experiments to ascertain the effect on its time of vibration of differences of temperature, by vibrating it in air of the natural temperature, and in air heated by means of boiling water. No memorandum has accompanied the observations of any accident having occurred, either in putting the needle away after the conclusion of the observations of the 29th, or before the commencement of those of the 30th, but a comparison of the results on the three days manifests that the magnetism of the needle sustained an alteration in that interval:

November 29. Corrected time of vibration 550·9 seconds
November 30. Corrected time of vibration 556·7 seconds
December 2. Corrected time of vibration 555·9 seconds.

The observations at the next station, San Blas, confirm this direct evidence of a change, as is seen in the following statement, which shows the quotients of No. 7. at Mazatlan and San Blas compared with the mean quotients of the other needles.

	1839.	Needle 7.	Mean of the Needles.	Difference.
Mazatlan	Nov. 29	0·934	0·933	+ 0·001
	Nov. 30 and Dec. 2	0·916	0·933	− 0·017
San Blas	Dec. 6 and Dec. 19	0·940	0·959	− 0·019

I have allowed, therefore, a loss of magnetism in this needle equivalent to 5″·4 in 556 seconds at all the stations subsequent to Mazatlan, and have accordingly deducted ·00848 from the logarithms of the squares of the times of vibration of No. 7. at those stations.

We now proceed to a similar general intercomparison of the needles at Martins Island.

Table IV.—Intercomparison of the Intensity Needles at Martins Island.

	Quotients.							
	5.	7.	8.	9.	11.	12.	13.	Mean.
Panama and Martins Island	0·975	0·983	0·981	0·981	0·981	0·983	0·978	0·980
Differences from the Mean	− ·005	+ ·003	+ ·001	+ ·001	+ ·001	+ ·003	− ·002	

Here, with the exception of No. 5, which appears to have sustained a slight loss of magnetism, the agreement of the quotients shows the general steadiness of the magnetic condition of the needles. In the case of No. 5. the difference does not exceed the limits of *occasional* error of observation; the results with this needle subsequently to San Blas may not deserve to be regarded as *fully* equal in value to those of the other needles; but the amount of error in the final determinations hazarded by retaining its results independent of correction is insignificant.

As this Table includes the whole interval between the observations at Panama in 1839, and those at Martins Island in 1840, we may regard it as substantiating the general steady magnetic condition of the needles in the whole of that interval, with the exception of the changes already noticed in Nos. 7. and 8, which have been traced to the period of their occurrence, and their amount examined and allowed for.

The times of vibration at all the stations visited subsequently to March 1839 were taken in arcs commencing with 20°: the time of the chronometer was noted at every 10th vibration during 300, and the mean time of 200 vibrations derived from ten partial results, i. e. from the 0th and 200th, the 10th and 210th, the 20th and 220th.

No satisfactory experiments having been made to determine the individual coefficients in the correction for temperature of these needles, I have taken an arbitrary coefficient for that purpose, being the arithmetical mean of the coefficients experimentally ascertained for the twenty-nine needles specified in the following list:—

·000165	Hansteen			Phil. Trans., 1828. Art. I.
·00019	Lenz	Needle	1	Mém. de l'Acad. Imp. de St. Pétersbourg, 1824.
·00025	Lenz	Needle	5	
·00026	Lenz	Needle	2	
·00029	Lenz	Needle	4	
·00016	Lloyd	Needle	L (4)	Trans. R. I. A., vol. xvii.
·000254	Lloyd	Needle	L (*a*)	Brit. Assoc. Report, 1835.
·000248	Lloyd	Needle	L (*b*)	

·00022	Sabine	Needle	L (3)	Account of the Euphrates Expedition. App.
·00027	Sabine	Needle	L (4)	
·00038	Sabine	Needle	E (1)	
·00041	Sabine	Needle	E (4)	
·000068	Sabine	Needle	FitzRoy's.	Voyage of the Beagle. App.
·00030	Sabine	Needle	(3)	Phil. Trans., 1840. Art. IV.
·000055	Ross and Sabine.	Needle	R L (4)	Brit. Assoc. Report, 1838.
·000436	Bache	Needle	. (1)	Trans. Am. Phil. Soc., 1836.
·000423	Bache	Needle	. (2)	
·000277	Bache . . .	Needle	. (3)	
·000117	Bache	Needle	. (A)	
·000052	Bache	Needle	. (C)	
·000357	Bache	Needle	(3 B)	
·000359	Christie . . .	Needle	. (1)	Phil. Trans., 1836. Art. XIX.
·000302	Christie . . .	Needle	. (3)	
·000177	Christie . . .	Needle	Lozenge	
·000227	Christie . . .	Needle	. II	
·00036	Duperrey . . .	Needle	. I	Mém de L' Acad. Roy. de Bruxelles, tom. xiii.
·000625	Quetelet . . .	Needle	. II	
·00020	Forbes . . .	Needle	. I	Trans. R. S. Edin., vol. xiv.
·00013	Forbes . . .	Needle	Flat	
·00026				

Whence,

$$T' = T\,[1 + \cdot00026\,(60° - t)],$$

in which T is the time of vibration at any station, t the actual temperature in degrees of Fahr., and T' is the equivalent time at the temperature of 60°.

The application of this correction gives the "corrected time" in Table V. In the few cases where the rate of the chronometer exceeded an insignificant amount, a correction for the rate is also included in the "corrected time," and a memorandum of the rate itself is inserted in the column of remarks.

Table V. contains an abstract of the observations at the different stations with the needles which have been specified: it includes every observation recorded to have been made with these needles between the 16th of March 1839 at Panama, and the 22nd of March 1840 at Bow Island, except, 1st, two incomplete observations, one with No. 11. at Fort Vancouver, and one with No. 13. at San Francisco, in which either the vibration was interrupted, or the needles came to rest, before the usual and requisite number of vibrations had been made; and 2nd, some observations at Tepic in the neighbourhood of San Blas and at Mazatlan, in which the needles, for the sake of experiment, were vibrated in air artificially heated, or alternately in the sun and in the shade.

TABLE V.—Abstract of observations with the Intensity Needles between the 16th of March 1839, and the 22nd of March 1840.

Station.	1839.	Needle.	Time.	Therm.	Corrected Time.	Remarks.
			s		s	
Panama	March 16.	5	472·6	84	469·4	
	March 16.	5	472·3	85		
	March 16.	7	535·2	80	531·3	
	March 16.	7	534·1	88		
	March 16.	8	470·2	88	466·2	
	March 16.	8	468·9	86		
	March 16.	9	437·8	80	433·8	
	March 16.	9	435·5	90		
	March 16.	11	473·3	87	470·0	
	March 16.	12	403·2	87	400·4	
	March 16.	13	393·2	88	390·4	
Cocos Island	April 8.	5	466·4	83	463·9	
	April 8.	5	466·9	82		
	April 8.	11	465·6	78	463·5	
	April 8.	11	465·8	78		
	April 8.	12	397·4	77	395·6	
	April 8.	12	397·5	78		
	April 8.	13	390·0	78	388·2	
	April 8.	13	390·0	78		
Oahu	June 4.	5	512·1	80		
	June 4.	5	513·8	90	510·3	
	June 9.	5	514·6	89		
	June 9.	5	515·2	91		
	June 8.	11	515·5	79	512·6	
	June 8.	11	515·8	88		
	June 8.	12	439·8	87	436·5	
	June 8.	12	439·4	88		
	June 8.	13	430·0	87	427·5	
	June 8.	13	430·9	87		
Kodiack	July 7.	5	688·5	79	685·1	
Sitka	July 18.	5	730·3	61	730·1	
	July 18.	11	730·4	62	730·1	
	July 18.	12	624·0	64	623·4	
	July 18.	13	611·2	67	610·1	
Woolwich	Aug. 12 and 13	7	766·4	66	766·2	Observed by Lieut.-Colonel SABINE.
	Aug. 12 and 13	7	766·8	66		
	Aug. 12 and 13	7	767·6	61		
	Aug. 12 and 13	8	670·7	68	669·7	
	Aug. 12 and 13	8	669·9	68		
	Aug. 12 and 13	8	671·7	63		
	Aug. 12 and 13	9	627·3	65	626·9	
	Aug. 12 and 13	9	627·8	61		
	Aug. 12 and 13	9	627·0	62		
Fort Vancouver	Aug. 13	5	616·8	59		Chron. G. 8ʰ·5
	Aug. 13	5	619·8	72		Chron. G. 8ʰ·5
	Aug. 14	5	616·9	65		Chron. G. 8ʰ·5
	Aug. 14	5	617·1	69	616·9	Chron. G. 8ʰ·5
	Aug. 14	5	619·8	69		Chron. G. 8ʰ·5
	Aug. 15	5	617·5	63		Chron. G. 8ʰ·5
	Aug. 15	5	617·2	63		Chron. G. 8ʰ·5
	Aug. 15	11	620·7	66	619·8	Chron. G. 8ʰ·5
	Aug. 13	12	527·5	82	526·9	Chron. G. 8ʰ·5
	Aug. 15	12	530·1	66		Chron. G. 8ʰ·5
	Aug. 13	13	515·6	81	515·2	Chron. G. 8ʰ·5
	Aug. 15	13	518·5	66		Chron. G. 8ʰ·5
Baker's Bay	September 13	5	622·4	60	622·6	
	September 13	5	624·4	70		

TABLE V. (Continued.)

Station.	1839.	Needle.	Time.	Therm.	Corrected Time.	Remarks.
			s	°	s	
Bodega	September 25 ..	5	560·4	63	559·9	
San Francisco	September 30 ..	5	556·8	76	555·0	
	September 30 ..	5	557·2	71		
	September 30 ..	11	558·7	64	558·1	
	September 30 ..	12	476·4	62	476·2	
Monterey	October 5	5	549·7	65	549·0	
Stª Barbara	October 10	5	539·0	77	536·6	
San Pedro	October 12	5	538·4	74	536·5	
San Diego	October 17	5	528·2	70	526·8	
San Quentin	October 24	5	515·3	77	513·1	
San Bartholomew	October 29	5	503·3	74	501·5	
Magdalena Bay	November 1 ..	5	490·1	73	488·5	
Bay of St. Lucas	November 21 ..	5	487·9	83	485·0	
Mazatlan	November 28 ..	5	488·1	71	486·7	
	November 28 ..	11	488·5	72	487·0	
	November 28 ..	12	416·9	72	415·6	
	November 28 ..	13	407·3	72	406·0	
	November 29 ..	7	552·8	73	550·9	
	November 30 ..	7	557·7	72	556·4	
	November 30 ..	7	559·4	73		
	December 2 ..	7	558·2	76		
	November 29 ..	8	485·1	72	485·3	
	November 30 ..	8	487·8	73		
	December 2 ..	8	488·1	76		
	November 29 ..	9	452·8	73	451·3	
	November 30 ..	9	453·3	71		
	December 2 ..	9	452·6	76		
	December 6 ..	5	482·3	88	479·5	
	December 19 ..	5	482·5	78		
	December 6 ..	7	550·3	85	548·4	
	December 19 ..	7	551·8	78		
	December 19 ..	7	551·8	78		
	December 6 ..	8	480·4	87	478·4	
	December 19 ..	8	481·9	77		
San Blas	December 6 ..	9	445·9	86	443·6	
	December 19 ..	9	446·2	76		
	December 6 ..	11	482·7	86	480·0	
	December 19 ..	11	483·1	81		
	December 6 ..	12	410·5	81	408·3	
	December 19 ..	12	411·0	84		
	December 6 ..	13	401·3	81	399·2	
	December 19 ..	13	402·0	84		
Socorro Island	December 26 ..	5	478·1	83	474·9	
	December 26 ..	5	477·7	85		
Clarion Island	December 29 ..	5	481·4	84	478·7	
	December 29 ..	5	482·0	85		
	1840.					
Martins Island (Marquesas)	January 23	5	477·5	88	475·4	
	January 23	5	477·9	89		
	January 28	5	479·2	85		
	January 28	5	480·1	86		
	January 29	5	479·0	86		
	January 25	7	545·0	86	541·0	
	January 27	7	544·6	88		
	January 25	8	476·0	87	472·5	
	January 27	8	475·7	83		
	January 25	9	441·5	90	438·1	
	January 27	9	440·8	86		

TABLE V. (Continued.)

Station.	1840.	Needle.	Time.	Therm.	Corrected Time.	Remarks.
			s	°	s	
Martins Island (Marquesas)	January 25	11	477·8	90	474·6	
	January 27	11	478·1	88		
	January 25	12	406·9	87	403·6	
	January 27	12	406·2	88		
	January 25	13	397·7	88	394·8	
	January 27	13	397·4	88		
	February 6 ..	5	484·1	87	480·5	
	February 6 ..	5	483·6	87		Village.
	February 22 ..	5	483·4	82	480·1	
	February 22 ..	5	483·5	89		Observatory.
	February 23 ..	5	485·6	90		
	February 25 ..	5	485·8	90	482·1	
	February 26 ..	5	485·0	81		S.E. Cocoanut Grove.
	February 26 ..	5	485·3	88	481·7	Chron. L. 5·3.
	February 26 ..	5	484·9	88		West Cocoanut Grove.
	February 27 ..	5	484·1	85	480·8	
	February 27 ..	5	484·0	83		West Rock.
	February 28 ..	5	484·4	86		
	February 29 ..	5	483·5	87	479·8	
	February 29 ..	5	482·8	81	481·4	Village.
	March 20 and 21	5	484·3	83		Mean of 22 sets. Observatory.
Bow Island	March 22.....	7	550·7	75		
	March 22.....	7	551·1	76	548·9	
	March 22.....	7	551·9	76		
	March 22.....	8	480·8	76		
	March 22.....	8	482·6	77	479·4	
	March 22.....	8	481·5	76		
	March 22.....	8	481·2	78		
	March 22.....	9	446·4	79		
	March 22.....	9	446·6	81	443·9	
	March 22.....	9	446·4	86		Chron. G. 5·8.
	March 22.....	11	482·8	87		
	March 22.....	11	483·1	82	479·5	
	March 22.....	11	482·6	87		
	March 22.....	12	411·0	86		
	March 22.....	12	411·1	87	408·2	
	March 22.....	12	411·2	88		
	March 22.....	13	402·2	87		
	March 22.....	13	401·9	87	399·1	
	March 22.....	13	401·8	85		

By means of the observations in the preceding Table, we obtain the ratio of the horizontal intensity at each station to that at each of the others specified in the Table. The *absolute* horizontal intensity was nowhere observed, because Captain BELCHER was not furnished with an instrument for the purpose, and no such instrument has yet been carried to any of the stations which he visited. For the purpose of expressing the ratio determined by the observations, we may select any one of the stations as a base-station, and assign an arbitrary value for the horizontal intensity at that station. I have chosen Panama, and have made the horizontal intensity there = 1000, because that is the value which it bears at Panama in M. GAUSS's theoretical map of this element*, and those who may desire it will thus be enabled to

* Atlas des Erdmagnetismus nach den elementen der theorie entworfen, Plate XI.; and Taylor's Scientific Memoirs, vol. ii. Plate XXII.

compare *directly* the horizontal intensities observed by Captain Belcher, with the computed intensities of M. Gauss's theory. Table VI. exhibits the observed values.

Table VI.—Observed Values of the Horizontal Intensity.

Station.	Needle.	Corrected Time.	Horizontal Intensity.		Number of		
					Needles.	Days.	Observations.
		s.					
Panama	5	469·4	1000				
	7	531·3	1000				
	8	466·2	1000				
	9	433·8	1000	1000	7	1	11
	11	470·0	1000				
	12	400·4	1000				
	13	390·4	1000				
Cocos Island	5	463·9	1024				
	11	463·5	1028	1022	4	1	8
	12	395·6	1025				
	13	388·2	1012				
Oahu	5	510·3	846				
	11	512·6	841	841	4	3	10
	12	436·5	841				
	13	427·5	834				
Kodiack	5	685·1	470	470	1	1	1
Sitka	5	730·1	413				
	11	730·1	414	412	4	1	4
	12	623·4	413				
	13	610·1	410				
Woolwich	7	766·2	480·9	480	2	2	6
	9	626·9	478·8				
Fort Vancouver	5	616·9	579				
	11	619·8	575	576	4	3	12
	12	526·9	577				
	13	515·2	574				
Baker's Bay	5	622·6	569	569	1	1	2
Bodega	5	559·9	703	703	1	1	1
San Francisco	5	555·0	716				
	11	558·1	709	711	3	1	4
	12	475·4	707				
Monterey	5	549·0	731	731	1	1	1
St^a Barbara	5	536·6	765	765	1	1	1
San Pedro	5	536·5	766	766	1	1	1
San Diego	5	526·8	794	794	1	1	1
San Quentin	5	513·1	837	837	1	1	1
San Bartholomew	5	501·5	876	876	1	1	1
Magdalena Bay	5	488·5	924	924	1	1	1
Bay of St. Lucas	5	485·0	937	937	1	1	1
Mazatlan	5	486·7	930				
	7	550·9	930				
	9	451·3	924	928	6	3	11
	11	487·0	931				
	12	415·6	928				
	13	407·3	925				
San Blas	5	479·5	959				
	7	548·4	957				
	9	443·6	956	958	5	2	10
	11	480·0	959				
	12	408·3	962				
	13	400·2	956				
Socorro Island	5	474·9	977	977	1	1	2

TABLE VI. (Continued.)

Station.	Needle.	Corrected Time.	Horizontal Intensity.		Number of Needles.	Number of Days.	Number of Observations.
Clarion Island....	5	478·7	962	962	1	1	2
Martins Island	5	475·4	975		7	5	17
	7	541·0	983				
	8	472·5	981				
	9	438·1	981	980			
	11	474·6	981				
	12	403·6	984				
	13	394·8	978				
Bow Island....	5	480·9	953		7	8	55
	7	548·9	955				
	8	479·4	953				
	9	443·9	955	957			
	11	479·5	961				
	12	408·2	962				
	13	399·1	957				

In this Table each needle has been given an equal influence on the mean result, without reference to the number of observations made with it. Where the observations do not afford certain and independent evidence of the unchanged state of each of the needles in respect to magnetism, weights assigned from other considerations must necessarily be arbitrary and uncertain. For example, at the last station in the Table, Bow Island, thirty-six observations were made with No. 5, and not more than three or four with each of the other six needles. But we have already seen, on intercomparison, reason to suspect that No. 5. may have sustained a slight loss of magnetism at the station preceding Bow Island, and it is the only needle in which any change of the kind is indicated subsequently to the general comparison at San Blas. Whilst, therefore, on the one hand, we might not be justified, without more clear and decided evidence, in altogether setting aside the result with No. 5, so on the other hand we should not obtain the most probable final deduction, by giving to that result a weight, in comparison with that of each of the other needles, proportioned to the number of observations, and resting on the probable error of a single observation,—apart from changes of magnetism in the needle itself.

We will now revert to the stations visited in the first voyage which were not subsequently revisited, and at which the values of the horizontal intensity may be derived by means of Nos. 5, 6, 7, or 8. Table VII. contains an abstract of the observations with these needles, in all of which the times of vibration were obtained in arcs commencing with 40°. The column entitled "Corrected Times," shows the mean time of vibration reduced to a standard temperature of 60°. The arithmetical mean of the times at Panama in March 1837, October 1839, and March 1839, has been taken as the approximate time of vibration at Panama throughout the interval; and the ratio of the horizontal intensity at the other stations has been computed accordingly, as shown in Table VIII. In this Table, as in Table VI., each needle

has been given an equal influence on the general mean, without reference to the number of observations which were made with it.

TABLE VII.—Abstract of observations with the Intensity Needles, Nos. 5, 6, 7, and 8, at the undermentioned Stations.

Station.	1837.	Needle.	Time.	Therm.	Corrected Time.	Remarks.
			s	°	s	
Panama	March 10	5	471·7	79	470·4	
	March 10	5	473·7	79		
	March 10	6	512·5	80	510·4	
	March 10	6	512·6	72		
	March 12	7	532·7	71	531·2	
	March 12	8	470·4	76	468·4	
Port Etches	August 28	5	736·7	50	738·7	Chron. L. 8ˢ·6.
	August 28	6	792·4	50	794·5	Chron. L. 8ˢ·6.
	1838.					
Acapulco	January 17	6	508·1	88	504·5	
	January 17	7	530·7	91	526·5	
Callao	June 20 and 21	5	487·1	68	486·1	Mean of 80 observations.
	June 25	6	525·0	79	522·4	
	June 27	7	546·1	72	544·6	
	June 27	8	483·8	70	482·6	
Puna Island (Guayaquil)	September 23	5	476·1	78	473·7	Mean of 90 observations.
	September 23	6	514·3	79		Pt Spanola.
	September 23	6	515·3	78		
	September 17	6	517·5	99		Pa Arena.
	September 17	6	517·4	99	510·5	
	September 19	6	517·1	93		Pt Barranca.
	September 20	6	509·7	86		Town of Puna.
	September 20	6	510·8	88		
Panama	October 28	5	475·3	82?	472·6	Therm. not recorded.
	October 28	6	514·6	82?	511·7	
	October 28	7	536·8	82?	533·7	
	October 28	8	471·4	82?	468·7	
	1839.					
Panama	March 16	5	476·4	84	473·2	
	March 16	5	476·1	85		
	March 16	7	536·8	80	532·9	
	March 16	7	535·3	88		
	March 16	8	471·8	88	468·1	
	March 16	8	471·0	86		

TABLE VIII.—Observed values of the Horizontal Intensity.

Station.	Needle.	Corrected Time.	Horizontal Intensity. Panama = 1000.	Remarks.
		s		
Port Etches	5	738·7	408	411
	6	794·5	414	
Acapulco	6	504·5	1026	1024
	7	526·5	1023	
Callao	5	486·1	943	950
	6	522·4	957	
	7	544·6	957	
	8	482·6	942	
Puna Island	5	473·7	993	998
	6	510·5	1002	

At four of Captain Belcher's stations in North America, he was preceded in observations of the horizontal intensity by Mr. David Douglas, who visited California and the Columbia River in the years 1830 to 1833. It may not be out of place to examine here the degree of accordance in the results obtained by the two experimenters at the four stations, Fort Vancouver, San Francisco, Monterey, and S^{ta} Barbara; and the comparison will be found instructive. Mr. Douglas's observations were made with two pairs of needles, which, before his departure for America, were vibrated in the environs of London, at intervals of several months, with consistent results. One pair of needles, numbered 3. and 4, were returned to England from San Francisco in 1831 to have their magnetic state re-examined: they arrived safely, and were vibrated in 1836, when, on a comparison with their rates in 1828 and 1829, No. 3. was found to have slightly gained, and No. 4. to have slightly lost magnetism; the consequence, probably, of their having been kept in constant contact with each other (No. 4. being a more powerful magnet than No. 3.), except when used in observation, when both needles were always vibrated, and their combined results considered as one determination. The mean of the times of vibration of these needles in 1828–1829, and in 1836, consequently furnishes a satisfactory London rate for the intervening years. The second pair of needles, numbered 5. and 6, were in Mr. Douglas's possession at the period of his untimely death at Owhyhee in 1834, as his letters contain the notice of observations made with them at the summit of Mowna Kaah, and in the crater of Kirauean, but they have not been found amongst his effects sent to England. The steadiness of this pair of needles can only be judged of, therefore, by their accordance everywhere with the results of Nos. 3. and 4.

Mr. Douglas's papers are in the Colonial Office; an account of his magnetic observations, which I drew up at the request of Lord Glenelg, then Secretary of State for the Colonies, was presented by His Lordship to the Royal Society, and was read in May 1837, but was not printed. The results of the horizontal intensity which will be now referred to, are taken from that account; they are also immediately deducible from the Table of the *total* intensities and dips observed by Mr. Douglas in North America, published in 1838 in my memoir on the magnetic Intensity of the Earth, in the Seventh Report of the British Association for the Advancement of Science: they are as follows:

Horizontal intensity: London = 1000.

	Nos. 5. and 6.	Nos. 3. and 4.
Fort Vancouver	1830 . . . 1238;	1830 . . 1220
San Francisco . .	1831 and 1833 . . . 1517;	1831 . . 1511
Monterey . . .	1831 and 1832 . . . 1566;	1831 . . 1542
S^{ta} Barbara	1831 . . . 1636;	Not observed.

or, if we regard London and Woolwich as identical in respect to the value of the horizontal intensity, and express this value by 480, which Captain Belcher's obser-

vations give as its ratio in August 1839 to 1000 at Panama, we have Mr. Douglas's determinations in immediate comparison with those of Captain Belcher as follows:

	Douglas, 1830, 1833.		Belcher, 1839.
Fort Vancouver	Nos. 5. and 6. 594;	Nos. 3. and 4. 586 . . .	576
San Francisco	Nos. 5. and 6. 728;	Nos. 3. and 4. 726 . . .	711
Monterey .	Nos. 5. and 6. 752;	Nos. 3. and 4. 740 . . .	731
S^{ta} Barbara .	Nos. 5. and 6. 785;	Nos. 3. and 4. not obs^{d}.	765

It has been assumed in this comparison that the horizontal intensity in London had the same representative value in the years to which Mr. Douglas's observations correspond as in the year to which Captain Belcher's correspond. But we know that the secular decrease of the dip in London causes a corresponding increase in the horizontal magnetic force at that station, and we are sufficiently acquainted with the average amount of the yearly diminution of the dip to introduce it as an element of calculation. Mr. Douglas's observations with Nos. 5. and 6. correspond to November 1828, when those needles were vibrated in London; and with Nos. 3. and 4. to January 1832, being the middle time between the observations before his departure, and those made with the same needles in June 1836, when returned to England. Captain Belcher's determination corresponds to August 1839, when his needles were vibrated at Woolwich. Taking the annual decrease of the dip in London in the interval at 2′·6*, and the value of the horizontal intensity at 480·0 in August 1839, we have its value 472·7 in January 1832, and 469·7 in November 1828; omitting the consideration of the secular change of the intensity itself, of which we know extremely little at present. Adopting these values of the horizontal intensity at the respective epochs, the American determinations become as follows, being all relative to 480 in August 1839.

	Douglas.		Belcher.
Fort Vancouver	Nos. 5. and 6. 581;	Nos. 3. and 4. 577 . .	576
San Francisco .	Nos. 5. and 6. 712;	Nos. 3. and 4. 714 . .	711
Monterey . .	Nos. 5. and 6. 736;	Nos. 3. and 4. 729 . .	731
S^{ta} Barbara . .	Nos. 5. and 6. 768;	Nos. 3. and 4. not obs^{d}.	765

There are still involved in the comparison the secular change of dip at the American stations, and the secular changes of the total intensity both there and in London: none of these are known sufficiently to make them proper elements of calculation, though we have reason to believe that the effect of each of these causes on the comparative numbers would be considerably less than that of the decrease of dip in London†. But enough has been said to show the large proportion which in such

* Eighth Report of the British Association, pp. 62. 66.

† By comparing Captain Belcher's observed inclinations with M. Hansteen's map of that element in 1780, we perceive that the inclination is annually increasing on the west coast of North America, but the amount of the annual change is apparently considerably less than that of the annual decrease in Europe:—with an annual increase of inclination we should have a decrease in the horizontal intensity; this corresponds with the remaining differences between the determinations of Captain Belcher and Mr. Douglas.

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determinations, the "corrections for epoch" and the uncertainties consequent thereupon, bear to the probable amount of the combined instrumental differences and observation errors of independent careful observers; and, consequently, the importance of the *synchronism* so much insisted upon, of the observations which are to concur in determining the magnetic state of the globe at the present period.

Inclination.—The inclinations contained in Table IX. were observed with a six-inch instrument by Robinson, with a needle by the same artist. The poles of the needle were reversed on every occasion, and the observations were repeated in the eight different positions of the circle and needle. Usually five repetitions were made in each position, and the arcs at both ends of the needle being read, the inclinations in the Table are generally a mean of eighty readings.

Table IX.—Observations of the Inclination.

Station.	Date.	Poles. Direct.	Poles. Reversed.	Inclination.		Remarks.
	1837.	° ′	° ′	° ′	° ′	
Panama	March 8.	30 58·8	32 53·2	31 51·0	31 51·9	Near the Ruins of the Convent of San Francisco.
	March 8.	30 54·6	32 51·0	31 52·8		
Magnetic Island	March 20.	30 27·5	32 01·5	31 14·5	31 11·9	
	March 22.	30 08·5	32 10	31 09·3		
Oahu	July 20 and 24.	40 43·4	42 27·6		41 35·1	
Port Etches	August 26 and 28	76 02·2	76 03·6		76 02·9	
Sitka	September 20.	75 47·5	75 53·8	75 50·6	75 51·5	Observation Island.
	September 22.	75 49·1	75 55·5	75 52·3		
San Blas	December 21.	44 20·2	46 30·2	45 25·2	45 24·3	Palm Island.
	December 22.	44 12·8	46 34·0	45 23·4		
	December 22.	43 59·2	45 26·3		44 42·7	Beach near the Arsenal.
San Francisco	December 29.	61 18·9	62 28·7		61 53·8	Yerba Buena.
	1838.					
Acapulco	January 18.	36 51·2	39 03·7		37 57·4	Near the Castle.
Realejo	March 19 and 21	33 13·6	36 00·2		34 36·9	Cardon Island.
Cocos Island	April 6.	22 29·2	24 36·9		23 33·2	
Callao	June 19 and 23	−4 20·9	−8 07·7		− 6 14·3	Plaza de los Muertes.
Puna Island (Guayaquil)	September 17.	8 09·4	9 11·1		8 40·2	Punta Arena.
	September 20.	8 58·1	10 15·8		9 36·9	Town of Puna.
	September 21.	8 22·3	10 27·9	9 25·1	9 08·0	P^t Spanola.
	September 24.	8 10·5	9 31·2	8 50·9		
	1839.					
Cocos Island	March 7.	21 26·6	24 24·9		22 55·7	
Oahu	June 1.	40 24·9	42 15·0	41 19·9	41 16·8	
	June 6.	40 12·5	42 09·7	41 11·1		
	June 7.	40 36·2	42 02·6	41 19·4		
Kodiack	July 7.	72 28·1	72 57·6		72 42·9	Near P^t Greville.
Sitka	July 19.	75 43·4	75 54·9		75 49·1	
Baker's Bay	September 13.	69 27·5	69 26·2		69 26·9	
Fort Vancouver	August 12.	69 15·9	69 23·6	69 19·8	69 22·2	
	August 12.	69 20·2	69 28·9	69 24·6		
Port Bodega	September 25.	62 50·6	62 56·2		62 53·4	On the Sandy Neck.
San Francisco	September 30.	61 50·2	62 16·0	62 03·1	62 05·0	
	September 30.	61 37·1	62 16·0	61 56·6		
	October 1.	61 52·3	62 25·7	62 09·0		
	October 1.	61 52·3	62 22·9	62 07·6		
	October 2.	61 50·2	62 25·7	62 08·0		

TABLE IX. (Continued.)

Station.	Date.	Poles. Direct.	Poles. Reversed.	Inclination.		Remarks.
	1839.					
Monterey	October 5	60° 59′·6	61° 07′·5		61° 03′·6	
Stª Barbara	October 10	58 53·2	58 55		58 54·1	
San Pedro	October 12	58 14·7	58 28·1		58 21·4	Fossil Island.
San Diego	October 17	56 49·5	57 22·7		57 06·1	
San Quentin	October 24	54 01·9	54 58		54 29·9	
San Bartholomew	October 29	51 12·7	52 09·4		51 41·0	
Magdalena Bay	November 1	45 22·1	47 40·5	46 31·3	46 34·0	
	November 1	45 37·5	47 36·0	46 36·7		
San Lucas Bay	November 21	44 37·5	46 41·0		45 39·3	
Mazatlan	November 29	45 42·7	47 30·0	46 36·3	46 38·5	
	December 2	45 43·6	47 37·7	46 40·7		
San Blas	December 6	43 34·5	45 22·8	44 28·7	44 32·5	Beach.
	December 19	43 40·8	45 31·7	44 36·3		
	December 8	43 39·9	45 39·4	44 39·6		
	December 8	43 38·4	45 39·4	44 38·9		
	December 10	43 42·7	45 29·9	44 36·3	44 38·3	Tepic.
	December 10	43 39·9	45 29·9	44 34·9		
	December 11	43 42·7	45 40·6	44 41·6		
Socorro Island	December 26	39 31·8	41 55·5		40 43·7	
Clarion Island	December 29	35 21·4	38 44·6		37 03·0	
	1840.					
Martins Island	January 23	−13 20·2	−14 53·6	−14 06·9	−14 06·0	Anna Maria Bay.
	January 25	−13 12·3	−14 50·6	−14 01·5		
	January 27	−13 22·5	−14 56·5	−14 09·5		
Bow Island	February 6	−30 13·9	−30 19·7	−30 16·8	−30 19·0	Entrance Village.
	February 29	−30 18·0	−30 24·4	−30 21·2		
	February 25	−30 07·1	−31 08·6		−30 37·9	S.E. point.
	February 26	−30 00	−30 38·2		−30 19·1	S.W. point.
	February 27	−29 43·4	−30 08·2		−29 55·8	Western extreme.
	February 22	−30 04·8	−30 13·4	−30 09·1	−30 15·3	Observatory.
	March 14	−29 44·6	−30 37·5	−30 11·1		
	March 20	−29 27·7	−31 11·1	−30 19·4		
	March 21	−29 28·7	−31 14·8	−30 21·7		

Total Intensity.—The values of the total intensity of Captain BELCHER's American stations are deducible from the values of the horizontal intensity in Table VI., and of the inclination in Table IX., by the formula

$$I' = I \cdot \frac{h' \sec i'}{h \sec i},$$

where i, h, and I are the inclination, horizontal and total intensities at Woolwich, and i', h', and I' the values of the same at any other station. Regarding Woolwich and London as identical, we have I = 1·372, the conventional number by which the total intensity in London is usually expressed. The values of the total intensity in Table X. have been thus computed.

Declination.—The declinations were observed with a nine-inch altitude and azimuth instrument by CARY, having a four-inch magnetic needle attached, which was read at both extremities. Each determination is stated to be the mean result of several observations, both of the true meridian, and of the magnetic direction.

TABLE X.—General Table of the Results of Captain BELCHER's Magnetic Observations on the West Coast of America, and the adjacent Islands. The longitudes in this Table are east of Greenwich; the declinations east; the values of the horizontal intensity are expressed relatively to 1000 at Panama; and the total intensities relatively to 1·372 at London.

Station.	Date.	Latitude.	Longitude.	Declination.	Inclination.	Intensity.		Remarks.
						Horizontal.	Total.	
Port Etches	1837	+ 60° 21′	213° 19′	31° 38·5	+ 76° 02·9	411	1·728	
Kodiack	1839	+ 57 20	207 09	26 43·5	+ 72 42·9	470	1·603	
Sitka	1837	+ 57 03	224 34	27 42·0	+ 75 51·5			
Sitka	1839	+ 57 03	224 38	29 32·5	+ 75 49·1	412	1·704	
Baker's Bay	1839	+ 46 17	235 58	19 11·0	+ 69 26·9	569	1·643	
Fort Vancouver ..	1839	+ 45 37	237 24	19 22·0	+ 69 22·2	576	1·657	
Port Bodega	1839	+ 38 18	236 58	15 20·0	+ 62 53·4	703	1·563	
San Francisco ..	1837	+ 37 48	237 37	15 20·0	+ 61 53·8			
San Francisco ..	1839	+ 37 48	237 37	15 20·0	+ 62 05·8	711	1·540	
Monterey	1839	+ 36 36	238 07	14 13·0	+ 61 03·6	731	1·531	
S^{ta} Barbara	1839	+ 34 24	240 19	13 28·0	+ 58 54·1	765	1·501	
San Pedro	1839	+ 33 43	241 45	13 08·5	+ 58 21·4	766	1·480	
San Diego	1839	+ 32 41	242 47	12 20·6	+ 57 06·1	794	1·482	
San Quentin	1839	+ 30 22	244 02	12 06	+ 54 29·9	837	1·461	
San Bartholomew	1839	+ 27 40	245 07	10 46	+ 51 41·0	876	1·432	
Magdalena Bay ..	1839	+ 24 38	247 53	9 15	+ 46 34·0	924	1·362	
Mazatlan	1839	+ 23 11	253 36	9 24	+ 46 38·5	928	1·370	
San Lucas Bay ..	1839	+ 22 52	250 07	8 37·5	+ 45 39·3	937	1·359	
San Blas.....	1837	+ 21 32	254 44	8 34	+ 45 24·3	.....	.....	Palm Island.
San Blas.....	1839	+ 21 32	254 44	9 00	+ 44 32·5	958	1·362	Beach.
Oahu Island	1837	+ 21 17	202 00	10 39·5	+ 41 35·1			
Oahu Island	1839	+ 21 17	202 00	.....	+ 41 16·8	841	1·134	
Socorro Island ..	1839	+ 18 43	249 06	.....	+ 40 43·7	977	1·307	
Clarion Island ..	1839	+ 18 21	245 19	.....	+ 37 03·0	962	1·222	
Acapulco	1838	+ 16 50	260 05	8 23	+ 37 57·4	1024	1·316	
Realejo	1838	+ 12 28	272 48	7 53·5	+ 34 36·9			
Panama	1837	+ 8 37	280 31	.....	+ 31 51·9	1000	1·193	
Magnetic Island..	1837	+ 8 04	278 15	7 37·5	+ 31 11·9			
Cocos Island	1838	+ 5 53	272 58	8 24	+ 23 33·2			
Cocos Island	1839	+ 5 53	272 58	.....	+ 22 55·7	1022	1·125	
Puna Island	1838	− 2 47	280 05	8 56	+ 9 08	998	1·024	
Martins Island ..	1840	− 8 56	220 20	.....	− 14 06·0	980	1·024	
Callao	1838	− 12 04	282 52	.....	− 6 14·3	950	0·968	
Bow Island.....	1840	− 18 05	219 07	.....	− 30 16·0	957	1·123	

The declination was observed by Captain BELCHER at Socorro, Clarion, Martin, and Bow Islands, but the record of the observations has not yet been sent home: an early opportunity will be taken of supplying this deficiency in the Table when the observations shall have been received.

§ 4. *New Determination of the Magnetic Elements at Otaheite.*

In M. Gauss's "General Theory of Terrestrial Magnetism*," there is a note to the following effect:—"Otaheite is a station of the highest importance for the future improvement of the magnetic elements: the difference between the two determinations of intensity made there by different observers, viz. Erman 1·172 in 1830, and FitzRoy 1·017 in 1835, is much greater than can with any degree of probability be attributed to yearly changes, and considerably exceeds the greatest difference between the computed and observed intensities at the eighty-six stations at which the theory has been compared with observations."

The importance which M. Gauss attached to a more exact determination of the magnetic elements at Otaheite was communicated to Captain Belcher in a letter from Captain Beaufort, which also conveyed to him permission from the Admiralty to touch at that Island on his homeward voyage. Quitting Bow Island in March 1840, Captain Belcher arrived at Otaheite in April, and made there the observations contained in Tables XI. and XIII., of which the results are given in Tables XII. and XIV.

Table XI.—Observations with the Intensity Needles at Otaheite.

1840.	Needle.	Time.	Therm.	Corrected Time.	1840.	Needle.	Time.	Therm.	Corrected Time.
		s	°				s	°	
April 4	5	483·9	87		May 6	8	478·9	88	
April 4	5	483·7	87		May 6	8	479·1	86	476·3
April 4	5	484·2	88		May 7	8	479·6	77	
May 3	5	482·8	90		May 7	8	478·6	76	
May 3	5	482·1	90		May 6	9	443·3	85	
May 4	5	480·8	74	479·5	May 6	9	443·4	84	441·6
May 4	5	482·3	86		May 7	9	444·2	75	
May 4	5	482·2	88		May 6	11	480·6	72	
May 4	5	482·2	89		May 6	11	480·4	80	478·0
May 5	5	481·6	76		May 6	11	480·7	89	
May 6	5	481·3	73		May 6	12	408·8	72	
April 17	5	480·1	77		May 6	12	409·1	85	406·8
April 17	5	480·6	77	478·2†	May 6	12	409·2	88	
May 6	7	547·9	94		May 6	13	399·8	73	
May 6	7	547·9	94	544·2	May 6	13	400·7	86	398·2
May 6	7	548·4	77		May 6	13	400·6	86	

* Resultate für 1838, I.

† Observed at Papeite: the other observations with No. 5, as well as all the observations with other needles, were made at Point Venus.

Table XII.—Observed values of the Horizontal Intensity at Otaheite.

	Needle.	Corrected Time.	Horizontal Intensity.	
		s		
Point Venus ..	5	479·5	958·6	
	7	544·2	971·8	
	8	476·3	965·1	
	9	441·6	965·1	965·4
	11	478·0	966·8	
	12	406·8	968·9	
	13	398·2	961·4	
Papeite	5	478·2	963·8	

Table XIII.—Observations of the Inclination at Otaheite.

1840.	Poles		Inclination.		Place of Observation.
	Direct.	Reversed.			
May 2	29° 48′·4	30° 54′·0	30° 21′·2		
May 2	29 46·7	30 52·7	30 19·7		
May 4	29 45·0	30 46·2	30 15·6	30° 17′·7	Point Venus.
May 5	29 48·0	30 43·9	30 15·9		
May 6	29 50·4	30 41·3	30 51·9		
April 11	26 17·0	28 01·0	.....	27 09	Papeite.

Table XIV.—General Results of Captain Belcher's magnetic observations at Otaheite.

Place of Observation.	Declination.	Inclination.	Intensity.	
			Horizontal.	Total.
Point Venus .	6° 30′ E.	+ 30° 17′·7	965	1·133
Papeite	not observed.	+ 27 09	964	1·098

Horizontal Intensity.—Captain Belcher's determination of the horizontal intensity falls between those of MM. Erman and FitzRoy.

Horizontal Intensity according to { Erman . . . . 1005 ; FitzRoy . . . . 874 ; Belcher . . . 965

But the differences of the three determinations far exceed in amount the errors, either instrumental or of observation, to which such experienced observers, provided with needles of steady magnetism, are liable; neither should we be justified in ascribing them to fluctuations in the magnetic force itself, especially at a station where the dip is small, and its variations have comparatively little influence on the horizontal intensity. The only known cause to which we can, with any degree of probability, attribute them, is to *station error*, in an island of which the basis is a volcanic rock: and to the same cause we must also refer the difference of three degrees in the inclination observed by Captain Belcher at Papeite and Point Venus, stations within seven geographical miles of each other, with an instrument of which the probable error, as derived from the observations at Point Venus, does not exceed the same number of minutes. In such localities it is well known to observers, that disturbing influences, producing differences as great as and even greater than those above stated, frequently occur at places not many yards apart. We know the intensity observations at Point Venus, with Captain Belcher's seven needles, to have been made at one spot, and it is worth while, therefore, to examine the degree of accordance with each other which their results present. Assuming the horizontal intensities determined by each of these needles to have an equal and independent value, and taking, therefore, the arithmetical mean as their most probable result, we have the errors of the needles as follows:—

No. 5.	Horizontal Intensity 958·6;	Error − 6·8
No. 7.	Horizontal Intensity 971·8;	Error + 6·4
No. 8.	Horizontal Intensity 965·1;	Error − 0·3
No. 9.	Horizontal Intensity 965·1;	Error − 0·3
No. 11.	Horizontal Intensity 966·8;	Error + 1·4
No. 12.	Horizontal Intensity 968·9;	Error + 3·5
No. 13.	Horizontal Intensity 961·4;	Error − 4·0
	Mean . . . . . 965·4	

Consequently the mean error ϵ_2* is

$$\epsilon_2 = \sqrt{\frac{117{\cdot}59}{6}} = 4{\cdot}425,$$

whence the *probable error*, r, of a determination with one needle is

$$r = \epsilon_2 . \rho \sqrt{2} = 0{\cdot}674489\, \epsilon_2 = 3{\cdot}0,$$

and the probable error of a determination with seven needles $\frac{r}{\sqrt{7}} = 1{\cdot}1$.

* Encke, Astron. Jahr. 1834, and Scientific Memoirs, vol. ii. Art. X.

M. Erman's determination differs 40·0, and Captain FitzRoy's 91·0 from Captain Belcher's; the former thirteen times, and the latter thirty times, the probable error of a determination with a single needle of steady magnetism, *where the spot of observation is the same.*

We have considered Captain Belcher's seven needles as giving *equal* and *independent* results for the ratio of the horizontal intensity at Otaheite and Panama; but the result with No. 8. is not strictly an *independent* one, inasmuch as at San Blas that needle received a small correction assigned from its comparison with the others; and the claim of the result of No. 5. to be considered as of *equal* value with that of each of the remaining needles is impaired by the probability of that needle having sustained a slight loss of magnetism at or before Martins Island (page 16.). If, therefore, we were to exclude the results of Nos. 5. and 8, and to derive the horizontal intensity at Otaheite from the five needles, which we may consider as of strictly independent and equal authority, we should have as their mean 966·8, with a probable error of 1·2, and the probable error of a determination with a single needle 2·6. It is true that the number of partial results from which this amount of the probable error is derived is small; but the probability of its being an approximately just representation of the errors of instrument and observation in this method, with needles of steady magnetism, is strengthened, if we examine in the same manner the results with the same five needles at the four stations preceding Otaheite; by so doing we obtain the probable error of a single needle from each as follows:—

Bow Island . . .	2·2
Martins Island . .	1·6
San Blas	1·5
Mazatlan	2·2

The integers in these quantities represent hundredth parts of the space comprised between two adjacent lines of horizontal intensity in M. Gauss's theoretical map of that element.

Uncertainties in respect to the magnetism of needles need no longer prove a source of vexatious anxiety and embarrassment even to travelling observers; with the simple apparatus described by M. Weber* the magnetic state of a needle may be examined at pleasure, and its magnetism may be altogether eliminated in the result.

With this advantage, however,—and it will be scarcely less valued by the confidence it creates whilst the observations are in progress, than by the independency it confers on their results,—and with a probable error of observation of extremely small amount, the magnetic traveller has still two serious sources of error to contend with: 1st, the values of the magnetic elements which he determines may not be *mean values*, by reason of the periodical or irregular fluctuations of the magnetic direction or

* Resultate für 1837, and Scientific Memoirs, vol. ii. Art. IV.

force possibly prevailing at the time of observation; and, 2nd, they may not be true measures of the magnetic elements corresponding to the geographical position in which the observations are made, by reason of those local disturbing magnetic influences which are included under the name of station error.

For the first of these sources of error a remedy is presented, whenever the observations can be made in connexion with those of a fixed magnetic observatory, situated within such distance that the magnetic elements are subject to the same periodical and irregular variations. The particular advantage possessed by the absolute determinations of the fixed observatories,—that of being *mean* values of the quantities sought,—may thus be indefinitely extended.

Against the more formidable evil of *station error* the connexion with a fixed observatory affords the magnetic traveller no security; nor can it furnish him with a correction,—for to error from this source the absolute determinations of fixed observatories are themselves no less liable; and no continuance, or frequency of repetition at the spot itself, will lead to its discovery or assign its correction. The magnetic survey of the British Islands, and more especially of its Scottish and Irish portions, has shown that such disturbances are not confined to localities, which, like Otaheite, consist chiefly of volcanic rocks, but may exist unsuspected and productive of error of serious amount, wherever the igneous rocks rise through, or approach the superficial soil. It is this source of error which presents a practical difficulty to the determination of the elements of the theory of terrestrial magnetism from exact observations at a few chosen positions on the globe. The remedy is to be found in the combination of fixed magnetic observatories and magnetic surveys; the observations of the survey being based on and executed in concert with the regular observations of a fixed observatory; the country surveyed being also sufficiently extensive to neutralise district anomalies, as well as those of a more local nature. The observations of the survey, corrected to mean determinations by their connexion with those of an observatory, and combined in the manner described by Mr. Lloyd in the third section of the Survey of the British Islands, will furnish in their turn the correction for the station error, if any, of the fixed observatory.

Total Intensity.—From the value of the total intensity at Otaheite as now determined by Captain Belcher, we learn that the *southerly inflection* of the isodynamic lines, in and about the meridian of the Society Islands,—which was pointed out as one of the characteristic features of the general configuration of those lines in the southern hemisphere*,—is even more strongly marked in the latitude of those islands than I had ventured to draw it, under the circumstances of the unusual discordance in the only observations which we then possessed.

* Seventh Report of the British Association, p. 73.

Declination.—I have collected in the subjoined Table all the recorded observations of the declination at Matavai Bay with which I am acquainted, from the earliest discovery of the island to the present time.

Table XV.—Declination observed at Matavai Bay, Otaheite.

Year.	Month.	Observer.	Declination. (East.)	Year.	Month.	Observer.	Declination. (East.)
			° ′				° ′
1765	June.....	Byron	5 00	1792	January	Vancouver	6 12
1767	July.....	Wallis	5 36	1823	May.....	Duperrey.....	6 40
1769	June.....	Cook	4 46	1824	March	Kotzebue.....	6 50
1773	August.....	Wales and Bayley	5 40	1826	April	Beechey	7 33
1774	April and May	Wales and Bayley	5 46	1835	November ..	FitzRoy	7 34
1777	December ..	Cook	5 34	1840	May.....	Belcher	6 30

A first glance at these observations shows that the easterly declination has been increasing at Otaheite from the time of the first discovery of that island. It is scarcely probable that the progression has been strictly uniform throughout the whole period, but the deficiency of determinations in the years that form the middle portion of the interval, renders the data that we possess unsuitable for deducing the variation in the rate of the secular change; and we must be content with that approximate representation which may be given by an uniform rate. Assuming, therefore, the change of declination to be proportional to the time, I have computed by the method of least squares from the data contained in the Table, the following formula for the declination δ at Otaheite:

$$\delta = 6^\circ\ 11'{\cdot}85 + 1'{\cdot}656\, t,$$

t being the interval of time elapsed since January 1, 1800, expressed in terms of a year.

The declinations computed according to this formula, and the differences from the observed declinations, are as follows:—

	Computed.	Differences.		Computed.	Differences.
	° ′	′		° ′	′
1765 . . .	5 15	+ 15	1792 . . .	5 59	− 13
1767 . . .	5 18	− 18	1823 . . .	6 51	+ 11
1769 . . .	5 21	+ 35	1824 . . .	6 52	+ 2
1773 . . .	5 28	+ 12	1826 . . .	6 55	− 38
1774 . . .	5 29	− 17	1835 . . .	7 12	− 22
1777 . . .	5 35	+ 1	1840 . . .	7 19	+ 49

It will be seen that the discordances with each other of the observations of recent date are as great, and even greater, than those of the earlier observers; which ought to be an indication that the larger discrepances are occasioned rather by local disturbing influences than by errors of observation. The probable error of a single determination, as resulting from the tabulated differences, would be about fifteen minutes.

Inclination.—The observations of the inclination made in the voyages of Captain Cook are entitled to much consideration, in respect both to the experience and skill of the observers, and to the goodness of their instruments. The English dipping needles of that period were made with much more care, and were much superior, especially in their axles, to those subsequently supplied to the government expeditions up to a very recent date. I have therefore placed in the subjoined Table the observations of Mr. Bayley in 1773, 1774, and 1777; and have combined them with the determinations of recent observers, for the purpose of exhibiting the secular change of the inclination at Otaheite, as deduced from the most unexceptionable data that we possess.

Table XVI.—Observations of the Inclination at Point Venus, Otaheite.

Year.	Month.	Observer.	Inclination.
1773	August....	Bayley ..	−29° 43′
1774	May.....	Bayley ..	−29 59
1777	December..	Bayley ..	−29 47
1823	May.....	Duperrey	−30 03
1830	September	Erman ..	−30 29·5
1835	November	FitzRoy	−30 14·5
1840	April	Belcher	−30 17·7

Whence, by the method of least squares, we obtain for the inclination the formula

$$I = -30^\circ\ 01'{\cdot}1 - 0'{\cdot}447\,t,$$

t being, as before, the interval of time elapsed since January 1, 1800, expressed in parts of a year. The inclination in January 1840, computed by the formula, is $-30^\circ\ 19'{\cdot}0$.

No observation recorded to have been made at Otaheite by Captain Belcher or his officers, has been omitted in the foregoing account: the manuscript records of the observations on the west coast of America and the adjacent islands, as well as those at Otaheite, are deposited in the Hydrographic Office of the Admiralty.

III. *On Ground Gru, or Ice formed, under peculiar circumstances, at the bottom of Running Water.* By James Farquharson, *LL.D. F.R.S., Minister of the Parish of Alford.*

Received March 11,—Read March 25, 1841.

In a paper of mine on Ground Gru, or ice formed at the bottom of running water, which was honoured with a place in the Philosophical Transactions*, I had inferred, from a great many conditions attending a remarkable occurrence of the phenomenon in the rivers Don and Leochal, in the beginning of January 1835, as well as from its occurring only when the air is at the time quite clear, that it is caused, when the water has gone down in temperature to the freezing point, by the bottom of the water being cooled to a still lower temperature, in the same manner as the surface of the dry land, under a clear sky, is cooled down below the temperature of the air, as first demonstrated by the experiments of Dr. Wells.

As the accuracy of the conclusion at which I arrived respecting the question has been controverted, I respectfully request the Royal Society to permit me to present to them brief notices of some recent occurrences of ground gru, in the same rivers to which I formerly referred, the conditions of which seem to me strongly to confirm the accuracy of the views I presented regarding the cause of the phenomenon; and also to answer some of the objections which have been brought against it.

Cold weather commenced on the 20th December 1840 (on which night the thermometer went down to 31°), and continued with frost every night, yet never below 26°, and with frost also through most of the day, till the 31st of the same month. By the 26th December, surface ice in considerable quantity was formed on the edges of the small river Leochal, and the temperature of the water was down to the freezing point. Down to the evening of the 28th the weather was cloudy, and there was no appearance in the river of anything resembling ground gru; but on that night the sky suddenly became clear; and before the morning of the 29th, the bottoms of all the rapids of the little river were thickly coated by the ground gru. The gru disappeared as speedily as it had formed, when, on the 29th, a close cloud, depositing slight showers of snow, again covered the whole sky, and continued till the temperature of the day and night rose above freezing.

In comparison with this, I would refer to a series of frosty days from the 1st to the 11th of February 1841, with a temperature the same as from the 22nd to 31st December, 1840, never descending below 26°. The water of the river descended to the freezing

* Part II. for 1835, p. 329.

temperature, and surface ice was formed in large quantity on the edges of both the Leochal and the Don. A dense cloud covered the sky during the eleven days and nights, and no ground gru appeared in the rivers.

A remarkable occurrence of ground gru took place in both the rivers from the evening of the 7th to the morning of the 9th January 1841, with a completely clear sky during the time. The thermometer was at 2° below zero on the night of the 7th, at 9° at mid-day on the 8th, and at 7° below zero on the night of the 8th. I examined particularly the state of the Don, during this extreme and clear frost, before it abated on the morning of the 9th. The bottom of the river was everywhere coated by an immense quantity of ground gru, excepting where it was partially shaded by bridges, or lofty banks close to the stream. In the partially shaded places the bottom was clear of gru. Thus this remarkable formation of ground gru took place under exactly such circumstances as those in which hoar-frost or dew takes place on the dry land, when the surface of the earth becomes colder than the air, (which we explain by a radiation of heat from the surface of the earth into the clear sky, or by impulses of cold from the sky to the earth,) with only this difference, that there was an additional transparent fluid over the bottom of the river, namely, the water; and thus also a shade prevented the formation of ground gru in the river, as it does that of hoar-frost or dew on the land.

In noticing the objections to the explanation I have given of the cause of ground gru, I shall confine myself to those brought forward by a writer in the Penny Cyclopædia, under the name of Ground Gru, which I have seen only very lately, although I believe they have been published for some years. He says the explanations of the formation of ground gru, given by Dr. Farquharson and Mr. Eisdale, are least of all satisfactory, and adds, "The former gentleman says it is the result of radiation, and endeavours to substantiate his reasoning upon the principles of the formation of dew, seeming to forget entirely, that Dr. Wells maintains expressly, that wind and shade are alike obstacles to radiation; and that consequently a body of moving water so deep as to be impervious to light, and particularly when covered, as in the case of the Neva, with a sheet of ice three feet thick, and as much more snow, must present an insurmountable obstacle to the radiation of heat from the bottom of the river."

Now, in the first place, with respect to shade; I was so far from forgetting that it is an obstacle to radiation, that, on the contrary, in my observations in 1835, I had shown by very many instances, that shade had prevented the formation of ground gru, just as it prevents dew. Wherever shade intervened to prevent radiation from the bottoms of the rivers Don and Leochal, there no ground ice was formed; while the unshaded parts of the bottoms were coated with it. My explanation thus mainly rested upon the fact that shade prevents radiation. In the next place, with respect to wind; the writer in the Cyclopædia himself forgets the difference of the statical conditions of air and water in connexion with temperature. Air becomes heavier by diminution of temperature. Water under 39° Fahr. becomes lighter by diminution

of temperature. During wind, on the land, the cold air at the surface of the earth is continually mixed with, or displaced by, the warmer air above; and by this process both the earth and air in contact with it are prevented from being reduced to a very low temperature by radiation. But in a body of moving water, whose temperature is under 39°, the eddies of the current throw down the coldest parts, which in still water would remain at the surface, to come into contact with the bottom. This last circumstance is the explanation of M. Arago; and it well accounts for the formation of ground gru taking place first in the most rapid parts of the streams; although neither by itself, nor when taken in conjunction with the other two circumstances to which he refers, namely, aptitude to formation of crystals on asperities at the bottom, and less impediment to the formation of crystals in a slower motion, will it account for the formation of ground gru, as all these circumstances are present when the water forms only surface ice. The formation of ground gru requires for its explanation an additional element, namely, the radiation, into the clear sky, of heat from the bottom of the river; and the formation never occurs but under a clear sky.

As to the ground gru, observed by Colonel Jackson in the Neva under three feet of ice and three feet of snow, that can form no valid objection to the explanation I have given, unless it were ascertained that the gru was formed after the surface ice and the fall of the snow, and not before them. All rivers issuing from lakes, like the Neva, have very clear waters to admit of radiation through them, although as deep as it is; and all rivers are very clear during frost, owing to the freezing up of the little land rills that would convey earthy particles into them. Ground gru formed in the Neva would be much more permanent than in our rivers. The mean temperature of Alford is 45° Fahr., and that of the earth of course the same; and on the remission of its cause, the ground gru is here speedily detached from the bottom, by the transmission of heat from below. Not so in the Neva. There, according to Kupffer, the mean temperature is only 38°·75; and under the action of a frost so severe as to form three feet depth of ice, although the condition of the clear sky might not continue, previously formed gru would, at that mean temperature, be of great permanence. This applies also to the Siberian rivers.

IV. *On a Remarkable Property of the Diamond.*

By Sir DAVID BREWSTER, *K.H. D.C.L. F.R.S. and V.P.R.S.Ed.*

Received February 15,—Read March 4, 1841.

HAVING had occasion, some years ago, to examine the structure of a diamond plano-convex lens which gave triple images of minute microscopic objects, I discovered, by a particular method of observation, that the whole of its plane surface was covered with hundreds of minute bands, some reflecting more and some less light; and I naturally drew the inference that this diamond consisted of a great number of layers of different reflective, and consequently refractive, powers, from which arose all its imperfections as a single microscope. In this case the veins or layers lay parallel, or nearly so, to the axis of the lens, so as to produce the worst effect upon the refracted pencil; for if the axis of the lens had been perpendicular to the surfaces of these veins, its performance as a microscope would scarcely have been injured by them.

In repeating Mr. AIRY's experiments on the action of the diamond in modifying NEWTON's rings near the polarising angle, I was led to re-examine the flat surface of the diamond above mentioned; but though I found my former observations perfectly correct, yet I was induced to suspect the accuracy of the inference which I drew from them, and which I could not but draw in the circumstances under which the phenomenon was presented to me.

In order that the Society may be able to judge of the new results at which I have arrived, I have given in Plate I. fig. 1. as accurate a drawing as I am able to make of the appearance of the flat surface of the diamond under consideration, as seen by light incident upon it nearly perpendicularly. The flat surface of the diamond is 0·058, or $\frac{1}{17}$th of an inch in diameter, and owing to the great convexity of its other surface, the light reflected by it does not interfere with the examination of the structure above mentioned.

The appearance shown in the figure is that which I observed some years ago; but upon shifting the line of illumination, I was surprised to perceive that *all the dark bands became light ones, and all the light bands became dark ones*, a phenomenon which placed it beyond a doubt that all *the bands were the edges of veins or laminæ whose visible terminations were inclined at different angles, not exceeding two or three seconds to the general surface.* Had this surface been an original face of the crystal there would have been nothing surprising in its structure, excepting the exceeding minuteness of the strata and the slight inclination of their terminal planes to each other; but being a surface ground and polished by art, the phenomenon which it presents is one extremely interesting.

The mineralogist will have no hesitation in admitting that this diamond is part of

a composite crystal consisting of a great number of individual crystals, like certain specimens of *feldspar, carbonate of lime*, and other minerals; but it is more difficult to conceive that the terminal planes of these individual crystals should retain their relative inclination after undergoing the operations of grinding and polishing upon a lapidary's wheel.

To many persons such a result may appear inadmissible; but there are several physical facts, which, when well considered, cannot fail to diminish its improbability. If we grind and polish a surface of *mother-of-pearl* obliquely to the strata of which it is composed, we shall find it impossible to produce a perfectly flat surface: even if we grind it on the finest and softest hone, and polish it with the smoothest powder, the termination of each stratum will remain; and while the general surface reflects a white image, the grooves or striæ will give rise to the beautiful prismatic images produced by interference*.

Another analogous fact presented itself to me many years ago in examining *calcareous spar*. Having had occasion to form an artificial face upon one of the edges of the rhomb containing the obtuse angle, I used a coarse file without water, and found that it exposed faces of cleavage which had never been previously seen, and which were inclined to the general surface produced by the file†.

In examining the optical figures produced by the disintegration of crystallized surfaces, I have found that by coarse sandstone, or the action of a rasp, or large-toothed file, we can expose surfaces of crystallization with their natural polish differently inclined to the general surface‡.

In all these cases the faces, exposed by the mechanical action of grinding or filing, preserve their natural surfaces and polish, and will preserve them more perfectly and readily if they are faces of easy cleavage. The facility of exposing such faces by the action of grinding must increase as the veins or strata become thinner, and it is probable that their exceeding minuteness in the diamond may have aided in the production of the structure which has been described.

I have found it quite impossible to measure the inclination of any of the faces by the goniometer; but I have succeeded, though with some difficulty, in taking an impression of the grooved surface upon wax.

This structure sufficiently explains the existence of three images when the lens was used as a microscope, without supposing that the veins had different refractive powers. Faces of different inclinations would, of course, converge the rays to different foci on the retina, as effectually as if there had been only a variation in their refractive indices.

* See Philosophical Transactions, 1814. † Edinburgh Journal of Science, Oct. 1828, vol. ix. p. 312.
‡ Trans. Royal Soc. Edin. vol. xiv.

St. Leonard's, St. Andrew's,
February 11th, 1841.

V. *On the Phenomena of Thin Plates of Solid and Fluid Substances exposed to Polarized Light.* *By Sir* David Brewster, *K.H. D.C.L. F.R.S. and V.P.R.S.Ed.*

Received April 15,—Read May 6, 1841.

HAVING received from Dr. Joseph Reade one of his beautiful instruments called the *Iriscope*, and made several experiments with it, I soon perceived that it might be advantageously employed in various investigations in physical optics. This instrument consists mainly of a plate of highly polished black glass, having its surface smeared with a solution of fine soap, and subsequently dried by rubbing it clean with a piece of chamois leather. If we breathe upon the glass surface, thus prepared, through a glass tube, the vapour is deposited in brilliant coloured rings, the outermost of which is black, while the innermost has various colours, or no colour at all, in proportion to the quantity of vapour deposited. The colours in these rings, when seen by common light, correspond with Newton's *reflected rings*, or those which have *black centres*, the only difference being, that in the plate of vapour, which is thickest in the middle, the rings in the iriscope have black circumferences*. By using a large system of rings, or depositing the vapour in straight lines in the plane of incidence, we can at once observe the phenomena of the coloured rings or bands at various angles of incidence.

The first person who investigated the modification of Newton's rings in reference to polarized light was M. Arago, who has given an account of his observations in a beautiful and highly interesting memoir, in the third volume of the *Mémoires d'Arcueil*, published in 1817. Without knowing what had been done by M. Arago, Professor Airy entered upon the same inquiry in 1831 and 1832; but the phenomena which he observed were the same as those which had been previously discovered by M. Arago, with the exception of the modification of the rings when formed by a lens pressed against the surface of a diamond.

When Newton's rings are formed by a lens pressed against a surface of glass, M. Arago observed that they were black centred, as usual; and whether viewed with the eye or with a doubly refracting rhomb of Iceland spar, that the single or the

* These rings may be formed upon almost all transparent bodies with more or less brilliancy, though I have found several substances, and occasionally pieces of glass, that will not absorb the soap. The rings are produced upon natural as well as artificial surfaces, that is, upon transparent surfaces produced by fusion or crystallization, as well as upon those polished by art. The soap being gradually dissolved by the vapour, requires to be frequently renewed. I find that other substances, particularly some of the oils, produce the same effect as soap. The rings disappear quickly by evaporation, and their brilliancy and purity of colour depend on the relative temperature of the vapour and the glass.

double system of rings had the same colours and the same diameters, the rings being completely polarized at the polarizing angle of the glass.

When the lens, however, was pressed against a metallic mirror, and examined with a doubly refracting rhomboid, two images perfectly similar appeared between a perpendicular incidence, and that of 55° or the polarizing angle of glass. One of the images disappeared entirely at this angle of 55°, when the principal section of the rhomboid was perpendicular or parallel to the plane of reflexion; but reappeared at greater incidences, with this remarkable peculiarity, that the colour of each of the rings which composed it was complementary to that of the corresponding rings in the image which had disappeared.

M. Arago likewise remarks that we may easily perceive with the eye, naked and without the assistance of any crystal, that at a certain angle near 55° the rings are composed of two distinct sets having unequal diameters, the rhomboid separating in a great measure the two sets of rings, because they are very unequally polarized. He likewise found that these phenomena were not produced when the rings were formed upon *native sulphur* and *diamond*.

"If the presence of a metallic mirror," says M. Arago, "is necessary for the production of the phenomenon in question when the rings are formed upon a plate of air, the case is otherwise when the thin body has much more density, and is in contact by one of its faces with another medium of sufficient refractive power. Thus *coal* presents often in its cleavages very bright colours, produced by an extremely thin substance, and which are decomposed into two complementary images when they are examined with a rhomboid under sufficiently oblique incidences. The colours which are formed artificially by the progress of evaporation, on thin films of alcohol or oil of sassafras, deposited upon coal or any other analogous substance, give rise also to two images, dissimilar, and of opposite tints*."

In order to investigate the phenomena of the rings of vapour in the iriscope, I illuminated them with light polarized in an azimuth of 90°, or perpendicularly to the plane of incidence, and examined them by a magnifying glass, when the centre of the rings was seen by light reflected at about 53° 11′, the polarizing angle of water. The effect, which was very striking, is shown in Plate I. fig. 2. The central part, A B, of the system of rings, C D E F, was without rings and colours of any kind: the upper half, C D, was part of a system of rings with *white* circumferences, and was formed by polarized light incident on the film at an angle *greater* than the polarizing angle of water; while the under half, E F, was part of a system of rings with *black* circumferences like those seen by common light, and was formed by polarized light incident on the film at an angle *less* than the polarizing angle of water.

The absence of rings in the middle portion, A B, was of course owing to there being no light reflected from the *first* surface of the film with which that reflected from the *second* surface could interfere; and the reason of there being light reflected

* Mémoires de Physique et de Chimie de la Société d'Arcueil, tom. iii. p. 363. Paris, 1817.

from the second surface was, that the light reflected from it was not incident at its polarizing angle.

I have elsewhere shown*, that when a film of water is laid upon glass whose refractive index is above 1·508, there is no angle of incidence upon the first surface of the film which will allow the refracted ray to fall upon the glass at the polarizing angle; and hence at every angle of incidence on the film, the refracted light is reflected from the glass *at angles less than the polarizing angle of the united media*, or less than an angle whose tangent is equal to $\frac{m}{m'}$, m being the refractive index of the glass, and m' that of the water. When the refractive index of the glass is 1·508, the angle of incidence on the film must be 90° exactly, in order that the refracted ray may fall upon the glass at the polarizing angle whose tangent is equal to $\frac{m}{m'}$.

Now as the portion of the coloured rings at C D, fig. 2., is formed by the interference of two pencils, C A, D E B, fig. 3., one of which, C A, is reflected at an angle, P C A, *above* the polarizing angle of water, and the other, E B, at an angle *below* or *less* than that angle; while the portion E F, fig. 2., is formed by the interference of two pencils, which are both reflected at angles *below* or *less* than that angle, we may suppose that in the formation of the rings with a white circumference, analogous to those with a white centre, there is a loss of half an undulation, while that loss takes place in the interference of common light, or of two pencils reflected on the same side of the polarizing angle.

When the rings are seen at angles between 0° of incidence and 53° 11′, the polarizing angle of water, they are *black* in the circumference, like the portion shown at E F, fig. 2.; and when they are seen at incidences between 53° 11′ and 90°, they are *white* in the circumference, like the portion shown at C D, fig. 2.

If the rings of vapour are formed upon a polished surface of *fluor spar*, additional phenomena will be exhibited. At all incidences, from 0° to about 78°, rings of the same character will be seen as already described; but the ratio of the refractive powers of water and fluor spar is such, that at an incidence of 78° 4′ upon the surface of the vapour, the light incident on the spar will be reflected at the polarizing angle of the united media. Thus if $m = 1{\cdot}437$, the refractive index of fluor spar, and $m' = 1{\cdot}336$, the refractive index of water, then $\frac{m}{m'} = 1{\cdot}0716$, the refractive index of the united media, or of their separating surface. The polarizing angle for this surface will therefore be an angle whose tangent is 1·0756 or 47° 5′, and the angle of incidence on the first surface of the watery film corresponding to the angle of refraction 47° 5′, which is the angle of incidence on the second surface, is 78°·4.

At an incidence of 78° 4′, therefore, the rings will disappear altogether, as at 53° 11′, because the pencil incident on the spar will not be reflected. At incidences greater

* Philosophical Transactions, 1815, p. 138.

than 78° 4′ the system of rings with the black circumference will again appear as at incidences below 53° 11′, and will be visible up to 90° of incidence, the interfering pencils being now both reflected at angles above the polarizing angle of the surfaces which reflect them.

This experiment with vapour and fluor spar I have not made; and it may be difficult to see the rings at such an oblique incidence. If the rings are formed by soap upon *plate glass*, or by *alcohol* upon *fluor spar*, the second disappearance of the rings may be seen:

$$\frac{\text{Plate glass}}{\text{soap}}\ \frac{m}{m'} = \frac{1{\cdot}510}{1{\cdot}487} = 1{\cdot}0154.$$

Polarizing angle at second surface of the soap . . . . 45° 26′
Angle of incidence on the first surface . . . . . 71° 45′

$$\frac{\text{Fluor spar}}{\text{alcohol}}\ \frac{m}{m'} = \frac{1{\cdot}437}{1{\cdot}370} = 1{\cdot}049.$$

Polarizing angle at second surface of alcohol . . . . . 46° 22′
Angle of incidence on the first surface . . . . . 82° 32′

If we call m, m' the indices of refraction of the two substances, viz. the *film* and the *surface* upon which it rests, m being the larger index, then a ray incident at 90° will fall upon the common surface of the two media at the polarizing angle of that surface, when the angle of refraction at the first surface is equal to the tangent, or cotangent of the polarizing angle, according as the refractive power of the film is less or greater than that of the body upon which it rests.

Hence we have $\sin i' = \frac{1}{m}$ or $\frac{1}{m'}$,

and

$$\tan i' = \frac{m}{m'}, \text{ or } \cot i' = \frac{m}{m'},$$

and

$$m = \frac{m'}{\sqrt{m'^2 - 1}}, \text{ and } m' = \frac{m}{\sqrt{m^2 - 1}},$$

when a ray incident at 90° is polarized at the second surface, or falls upon it at the polarizing angle.

These formulæ enable us to discover between what limits of refractive power the second disappearance of the rings can take place, and consequently what substances we should employ in order to observe it. In this manner we obtain the following results for the mean rays of the spectrum:—

Values of m'.	Values of $\frac{m'}{\sqrt{m'^2 - 1}}$, or m.
3·000	1·061
2·500	1·090
2·000	1·154
1·900	1·176

Values of m'.	Values of $\frac{m'}{\sqrt{m'^2-1}}$, or m.
1·800	1·202
1·700	1·236
1·600	1·281
1·554	1·307
1·508	1·336
1·500	1·341
1·400	1·428
1·336	1·508
1·307	1·554

The limits, therefore, between which the *second* disappearance of the rings can take place are 1·554, the index for *quartz* and *flint glass*, and 1·307, the index for *ice*. But though the range is very limited, it nevertheless includes a considerable variety of solid and fluid bodies. I have omitted the indices of Tabasheer, and the fluids produced by the compression of gaseous bodies, because, though their refractive powers are beneath 1·307, they cannot be used in the present inquiry.

When m and m' are thus related, the *white centred rings* will just disappear when $i = 90°$, the light being then incident on the second surface at its polarizing angle. But if we use a film of still less refractive power in relation to the second body, the refracted rays will fall on the second surface at an angle *greater* than the polarizing angle (i being still 90°), and consequently *the black centred rings will reappear*, and there will be some angle of incidence I on the film, less than 90°, at which the angle of refraction i' will be equal to the polarizing angle of the second surface. This angle will be found from the expression

$$\sin \mathrm{I} = \frac{m\,m'}{\sqrt{m^2 + m'^2}}.$$

When $m = m'$ no rings whatever will be formed, as no light is reflected at the common surface; but if $m = m'$ only for a particular colour in the spectrum of each substance, and if these indices differ considerably for another colour, rings will be formed in which that colour predominates, in which $m > m'$, or $m < m'$. This takes place in a remarkable manner with *oil of cassia* and *flint glass*, in which $m = m'$ for the *red* rays, but $m > m$ for the *blue* rays. The consequence of this is, that a quantity of *blue* light is reflected from the separating surface of the oil and the glass; and hence if a sufficiently thin film of oil of cassia is laid upon the glass, *blue* would greatly predominate in the system of rings.

Hitherto the azimuth of the polarized light has been 90°, or perpendicular to the plane of reflexion. Let us now suppose that its azimuth is gradually changed from 90° to 0° by the rotation of the polarizing surface or crystal.

At all azimuths, from 90° to 0°, the rings with the black circumference are seen, between the angles of 0° and 53° 11′, and at the incidence of 53° 11′. But at inci-

dences between 53° 11′ and 90°, in the case of the iriscope, very interesting phenomena appear. We shall first describe what takes place at 56° 45′, the polarizing angle of the black glass. At this angle none of the polarized light is reflected when the azimuth is 90°, and the rings with the *white* circumference are beautifully seen on the dark ground of the glass, which now reflects no light. As the azimuth is changed to 87°, 88°, &c., the black glass reflects a little light, and the two surfaces of the film a little more light, the rings gradually become fainter and fainter, till at an azimuth of about 79° 0′ they disappear exactly as they did at 53° 11′, and in the azimuth 90°. When this disappearance takes place, the light reflected from the glass seems to be exactly equal to the light reflected from both surfaces of the film. At other angles of incidence the rings disappeared at different azimuths, varying from 90° to about 45°, as the angle of incidence varied from 53° 11′ to 90°. I found it difficult, however, to measure these azimuths with any accuracy, as the rings were not permanent; and I was therefore obliged to form the colours of thin plates upon highly refracting substances, such as *diamond*, *chromate of lead*, *artificial realgar*, and *greenockite* (the most refractive of all bodies), which had high polarizing angles. A solution of fine soap gave brilliant colours when dried, and in this way I obtained the following results with the surface of a very fine diamond. The index of refraction of the soap was 1·475, and that of the diamond 2·44, and their respective polarizing angles 55° 52′, and 67° 43′.

Angle of Incidence of the Polarized Light.	Azimuth of the Plane of Polarization at which the rings disappear.	
	Observed.	Calculated.
55° 52′	90° 0′	90° 0′
60	73 0	74 27
65	68 30	67 49
67 43	66 20	65 10
70	63 30	63 14
75	59 15	58 23
90		46 30

As the disappearance of the rings was not owing to the extinction of one of the interfering pencils, as at 55° 52′, for a sufficient quantity of polarized light was reflected from both surfaces of the film, there was reason to believe that it might arise from the two pencils being polarized at right angles to each other, in conformity with the law relating to the action of the second surfaces of plates which I have given in a former paper*.

Calling x the azimuth of primitive polarization, i the angle of incidence on the *first* surface of the film, i' the corresponding angle of refraction, and consequently the angle of incidence on the *second* surface, i'' the angle of refraction at the *second* surface,

* Philosophical Transactions, 1830, pp. 148, 149.

and ϕ = the inclination of the plane of polarization of the reflected pencil C A, fig. 3.
ϕ' = that of the refracted pencil C D,
ϕ'' = that of the reflected pencil D E, and
ϕ''' = that of the refracted pencil E B, with which C A interferes; then by Fresnel's formula we have for the ray C A,

$$\tan \phi = \tan x \cdot \frac{\cos (i + i')}{\cos (i - i')};$$

and by my formulæ* we have

$$\cot \phi' = \cot x \cos (i - i')$$

$$\tan \phi' = \tan x \cdot \frac{1}{\cos (i - i')}$$

$$\tan \phi'' = \tan x' \cdot \frac{\cos (i' + i'')}{\cos (i' - i'')}.$$

But, after one refraction,

$$\tan x' = \tan \phi = \tan x \cdot \frac{1}{\cos (i - i')};$$

hence

$$\tan \phi'' = \tan x \cdot \frac{1}{\cos (i - i')} \cdot \frac{\cos (i' + i'')}{\cos (i' - i'')}$$

and

$$\cot \phi'' = \frac{1}{\tan x} \cdot \cos (i - i') \cdot \frac{\cos (i' - i'')}{\cos (i' + i'')}.$$

And multiplying this by $\cos (i - i')$ for the change of plane produced by the second refraction at E, we have for the ray E B,

$$\cot \phi''' = \cot x \cos^2 (i - i') \cdot \frac{\cos (i' - i'')}{\cos (i' + i'')}.$$

Now the two pencils which interfere, viz. C A and E B, have their planes of polarization inclined at angles ϕ and ϕ''' to the plane of reflexion; but in order that these angles may be complementary to each other, or may together make 90°, we must have $\tan \phi = \cot \phi'''$, or

$$\tan x \frac{\cos (i + i')}{\cos (i - i')} = \cot x \cos^2 (i - i') \cdot \frac{\cos (i' - i'')}{\cos (i' + i'')};$$

and consequently

$$\tan^2 x = \cos^2 (i - i') \cdot \frac{\cos (i - i')}{\cos (i + i')} \cdot \frac{\cos (i' + i'')}{\cos (i' - i'')};$$

and

$$\tan x = \cos (i - i') \cdot \sqrt{\left(\frac{\cos (i - i')}{\cos (i + i')} \cdot \frac{\cos (i' - i'')}{\cos (i' + i'')}\right)}.$$

When the angle of incidence is 90°, $\cos (i + i') = \sin i'$, and $\cos (i - i') = \sin i'$, and hence

$$\tan x = \frac{1}{m} \sqrt{\frac{\cos (i' - i'')}{\cos (i' + i'')}}.$$

* Philosophical Transactions, 1830.

If we now calculate by these formulæ the values of x for the different angles of incidence in the preceding Table, and subtract them from 90°, we shall have the numbers in the third column of the Table, which agree with those observed within the limits of the errors of observation. In the case of *water* and *glass*, too, where the azimuth of disappearance was observed to be about 79° or 11°, the formula gives 79° 28′, or 10° 32′, at an incidence of 56° 45′.

In order to ascertain the relation between the mutual inclination of the planes of polarization of the interfering pencils when they produced *black-centred* or *white-centred* rings, I have computed the following Table for an incidence of 56° 45′.

Azimuth of Polarized Light.	$+\phi$	$-\phi'''$	Film of *water* and *glass*. Inclination of Planes ϕ and ϕ'''.	
90° 0′	90° 0′	90° 0′	180° 0′	*White*-centred rings.
87 30	74 43	82 45	157 28	
85 0	49 30	75 4	124 34	
79 28	28 26	61 34	90 0	*No rings.*
70 0	15 28	43 19	58 47	*Black*-centred rings.
45 0	5 45	18 57	24 42	
35 0	4 3	13 3	17 6	
20 0	2 6	7 7	9 13	
0 0	0 0	0 0	0 0	

By taking φ *positive*, or on the *right*-hand side of the plane of reflexion, then φ''' must be *negative*, or on the left-hand side of that plane*; hence $+\varphi, -\varphi'''$ will be the mutual inclinations of the planes of polarization of the interfering pencils, and we obtain the important law,

That when two polarized pencils reflected from the surfaces of a thin plate lying on a reflecting surface of a different refractive power interfere, half an undulation is not lost, and WHITE*-centred rings are produced, provided the mutual inclination of their planes of polarization is greater than* 90°; *and that when this inclination is less than* 90°, *half an undulation is lost, and* BLACK*-centred rings are produced; when the inclination is exactly* 90°, *the pencils do not interfere, and no rings are produced.*

At an incidence of 45° upon water and glass, where the signs of φ and φ''' are the same, the maximum difference in the planes of polarization is 23° 12′, which takes place in azimuth 70° 30′; and at an incidence of 10° the greatest difference is 2° 16′, which takes place at an azimuth of about 45°.

In the case of *soap* and *plate glass*, where the black-centred rings appear beyond the incidence of 71° 45′, the difference of inclination in the planes of the two pencils is also less than 90°.

I was now desirous of examining the phenomena of a perfect system of rings when the film had a greater refractive power than the substance upon which it was laid;

* See Philosophical Transactions, 1830, p. 70, fig. 1.

after many ineffectual attempts to obtain such a system, I succeeded by laying a very small portion of *oil of laurel* upon *water* placed in a black vessel, or on the surface of diluted or real ink. The rings thus produced are splendid beyond description, and exhibit the various phenomena with singular beauty. As the polarizing angle of the oil *exceeds* that of the water, the *black*-centred rings are seen at the polarizing angle of the water, when the reflected light disappears. They continue to be seen till we reach the polarizing angle of the oil, when the rings disappear, and the white-centred ones commence, and continue till we reach the incidence of 90°*.

In forming thin films upon metallic surfaces, I employed many of the metals, and found the phenomena nearly the same upon them all, and differing very little from those produced upon transparent bodies. On a fine specimen of *specular iron ore*, I found a system of rings ready formed, with three orders of colours. The azimuth of the polarized light being inclined 90° to the plane of reflexion, the system of rings disappeared wholly at an angle of incidence of 58° 36′, which is therefore the polarizing angle of the unknown substance of which it was formed: consequently its index of refraction is about 1·638. Between this angle and 90° of incidence, the *white*-centred rings appeared; but at 72° 39′, the polarizing angle of the iron (which gives its refractive power for the *red* rays 3·200), the rings were singularly fine, being seen on a beautiful blue ground, produced by the disappearance of the *red* light, which is polarized at that angle. I now measured the azimuth of the plane of polarization when the rings disappeared, which was 59° 25′, whereas by the formula it is 57° 59′; a discrepancy not to be wondered at, when we consider that the index of refraction for the red rays, viz. 3·200, was used, in place of that for the mean ray, which is not known. The inclination of the planes of polarization of the two interfering pencils, when calculated by the previous formulæ, is + 32° 7′, and − 57° 53′; so that these planes being inclined 90° to each other, as in the case of soap and diamond, no interference takes place, and the rings disappear.

In the fine specimens of *oligist iron ore* from Elba, I have found crystals covered with the most beautiful coloured films, both of uniform and variable thickness. These films are not acted upon by the ordinary acids, like the coloured films upon steel, and appear, from their optical properties, to be of a metallic nature. When they are exposed to a polarized ray, they exhibit generally the same phenomena as the films already described; but there is no angle of incidence at which the colours disappear, either in the azimuth of 90°, at the polarizing angle of the first surface of the film,

* These thin plates of oil of laurel exhibit some curious phenomena, which I believe have not been noticed. If we wet with water, alcohol, or the oil of laurel itself, the extremity of a short piece of wire, such as a large pin, and hold the pin in the hand, so that its head may be above, and almost touching the film, the film will recede in little waves of a circular shape, which form a new system of coloured rings; and they become covered with the vapour from the fluid on the head of the pin in such small particles that they reflect no light, and the rings appear to be blackened. By withdrawing the pin, the film is restored to its former state. The same effect is produced by heating the pin, or the fluid upon it, to promote evaporation.

or in those azimuths where the pencils, from the first and second surface, have their planes of polarization inclined 90° to each other. This, no doubt, arises from the high dispersive power of the film, in consequence of which the different homogeneous rays are polarized at angles considerably different from each other.

The phenomena of transparent films of low refractive power, when laid upon the polished surfaces of metals, and exposed to polarized light, are not very different from those which are exhibited when the film rests upon a transparent surface. I at first used a solution of soap, which produced pretty good tints on speculum metal; but at last I fell upon a method of laying down the most beautiful systems of coloured rings upon all surfaces of all forms, whether metallic, transparent, or opake. For this purpose I used the *oil of laurel*, which, when placed upon the surface of water, expands into a film, which gives the finest system of coloured rings. Having laid the plate of polished metal in a small porous wooden tray, such as is used for holding minerals, I poured water into it, so as to cover the metallic surface to the depth of the 50th part of an inch. I then formed a film of the oil upon the water, immediately above the metallic surface. In a short time the absorption of the water by the porous tray allowed the film of oil to descend and rest upon the metallic surface*. When the adhering moisture was removed by evaporation, the film was extremely beautiful; and if protected from dust may be preserved for any length of time.

Having laid a film of this kind upon *speculum metal*, I obtained the following results. The coloured rings disappeared almost completely at 56°, the polarizing angle of the oil. The *black-centred* rings appeared at all angles less than 56°, and the *white-centred* rings at all angles above it. Both the systems of rings were exceedingly distinct at the greatest angles of incidence, whereas on transparent surfaces of low refractive power, they can scarcely be seen at such angles. When the azimuth of the polarized ray varies from 90° to 0°, the rings disappear at different angles of incidence; or when the angles of incidence vary, the rings disappear in different azimuths. I measured these azimuths when the polarized ray was incident upon speculum metal, and obtained the following numbers:—

Angles of Incidence.	Azimuth in which the rings disappear. Observed.	Calculated.	Difference.
° ′	° ′	° ′	
90		40 23	
71 50	56 25	57 22	− 0 57
60 0	65 45	65 4	+ 0 41
56 5	90 0	90 0	

In computing column third from the formula in p. 49, I used 1·49 as the index of refraction of *oil of laurel*, and 4·011 as the index of refraction for *speculum metal*, as deduced from my experiments on its elliptic polarization†.

* The same effect is produced more slowly by evaporation; or the water may be sucked out of the tray by a tube, or run off by an aperture.

† Philosophical Transactions, 1830, p. 324.

I have made similar experiments when the rings were transferred to *silver*, whose elliptical polarization approaches nearest to circular polarization; and to *grain tin*, which appears to have the highest refractive power of any of the metals; but I found it very difficult to ascertain with any accuracy the azimuths in which the rings disappear.

If we use common in place of polarized light in the preceding experiments, and analyse the reflected light by a rhomb of calcareous spar, the very same phenomena will be exhibited.

When the films or thin plates are not laid upon the surfaces of fluid or solid bodies, the phenomena are of an entirely different kind. At all angles of incidence, and in all azimuths, the colours and character of the rings are the same, whether we use common or polarized light. In obtaining this result I stretched thin films of various oils, such as *oil of laurel*, *oil of cassia*, *oil of turpentine*, and many others, across circular apertures, and examined them in light polarized in different azimuths. The rings of course vanished at the polarizing angle of the oil, and the brilliancy of the colours varied with the angles of azimuth and incidence, but the complementary rings never appeared, the rings being always those with the black centre*.

In order to understand the cause of this, we must inquire into the state of polarization of the interfering pencils. The ratio of refraction being the same at both surfaces of the film, we have

$$\tan \varphi = \tan x \cdot \frac{\cos (i + i')}{\cos (i - i')}, \text{ and } \cot \varphi''' = \cot x \cdot \frac{\cos^3 (i - i')}{\cos (i + i')};$$

and when $\tan \varphi = \cot \varphi'''$, which is the case when $\varphi + \varphi''' = 90°$, we have

$$\tan x = \frac{\cos^2 (i - i')}{\cos (i + i')}.$$

When $i = 90°$, $\tan \varphi = A$, or the azimuth of the polarized ray, and $\cot \varphi''' = \frac{\cos^3 i'}{\sin i'}$.

If we now compute the values of φ and φ''' at different angles of incidence and in different azimuths of the polarized light, we shall obtain the results in the following Table. In azimuths 0° and 90°, φ and $\varphi''' = 0$.

* The physical phenomena exhibited in these attenuated films are very remarkable. A current of fluid is projected from the margin and centre of the ring of fluid across the fluid surface, resembling the top of a pine apple. This movement makes the film thinner at some places than others, and hence arises an irregular system of coloured bands, with an incessant play of varying tints, as if the fluid were animated. The bands of colour are serrated with salient points, from which the fluid seems to shoot across the film. In the oils of cinnamon, naphtha, spearmint, wormwood, rapeseed, nutmegs, bergamot, savine, rosemary, &c., the phenomena are peculiarly beautiful. With poppy oil, the *red* and *green* tints of the 4th, 5th, and 6th orders were also seen.

Angles of Incidence.	Inclination of the planes of polarization of the two pencils, ϕ and ϕ'''.							
	Azimuth 22° 30′.		Azimuth 45°.		Azimuth 67° 30′.		Azimuth 80°.	
	Pencil from *first* surface.	Pencil from *second* surface.	Pencil from *first* surface.	Pencil from *second* surface.	Pencil from *first* surface.	Pencil from *second* surface.	Pencil from *first* surface.	Pencil from *second* surface.
	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0	22 30	22 30	45 0	45 0	67 30	67 30	80 0	80 0
10	21 42	22 5	43 51	44 24	66 40	67 4	79 36	79 48
20	19 11	19 34	40 13	40 38	64 13	64 14	78 13	78 23
30	15 25	15 55	33 40	34 33	58 7	58 58	75 10	75 38
40	10 18	11 1	23 41	25 11	43 21	48 37	68 6	68 6
50	4 18	4 52	10 18	11 37	23 41	26 24	45 52	49 23
56 in 45	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0
60	2 6	2 35	5 4	6 13	12 5	14 44	26 42	31 42
70	7 54	11 52	18 32	26 53	39 0	50 45	62 16	70 49
80	15 11	24 41	33 13	47 58	57 41	69 32	74 56	80 58
85	18 40	33 34	39 12	58 2	63 5	75 30	77 48	83 43
90	22 30	43 57	45 0	66 44	67 30	79 54	80 0	85 40

The results in this Table, which may be considered as those of observation*, exhibit at one glance the general phenomena at all angles of incidence and azimuth.

The two interfering pencils are in every case reflected at angles either *both above* or *both below* the polarizing angle, and hence their planes of polarization are always on the same side of the plane of reflexion and in the same quadrant, and consequently they never can be at right angles to each other so as to prevent interference. For the same reason the inclination of the planes never can exceed 90°, so as to produce the complementary white-centred rings, in conformity with the law previously given.

If, for example, we compute the value of x in the preceding formula at an incidence of 70°, we shall find it 66° 25′, at which azimuth the inclinations ϕ and ϕ''' of the planes of polarization are 40° 47′, and 49° 53′; but though the sum of these angles is 90°, yet the real inclination of the planes is $\phi''' - \phi = 9° 6'$.

This property of parallel transparent films, of giving by reflexion pencils polarized in planes at various inclinations, when the incident light is polarized in different azimuths, enables us to obtain two pencils of polarized light, inclined at any angle, varying from 0° to 21° 44′ in glass, and to study the phenomena which such pencils exhibit, either in their mutual action, or in their relations to other properties of light.

But the phenomena become more varied and interesting when the second surface of the plate is *inclined* to the first. In this way we may produce effects analogous to those produced by a change in the refractive power of the second surface by contact with another refracting surface, and obtain pencils inclined 90° to each other, and therefore exhibiting the white-centred rings. The phenomena will in this case resemble those of a film of oil upon water.

When the refractive index of a parallel film exceeds 1·508, the ray is incident on the second surface at an angle less than the polarizing angle; but by inclining the

* See Philosophical Transactions, 1830, pp. 74, 138.

second surface we can make it fall upon it at a greater angle than the polarizing angle. The phenomena may be still more varied by inclining the surface of emergence to the surface of incidence*; but as it is not easy to obtain films with faces suitably inclined to each other, it is unnecessary to pursue this branch of the subject any further.

Such are the phenomena of *thin* and *thick* plates when viewed by polarized light, or by common light subsequently analysed by a doubly refracting rhomb. But if we use polarized light, and subsequently analyse the light transmitted through the thin plates, we shall obtain a series of very interesting and instructive phenomena, analogous to those produced by plates of doubly refracting crystals which exhibit the polarized tints. In both these cases, the film is interposed between a polarizing plate and an analyzing rhomb. If the film is too thick to produce colours, it will depolarize the polarized ray, in a manner analogous to that of a crystallized plate, which is not thin enough to give the polarized tints; and if the film is sufficiently thin to produce uniform tints, a coloured band or system of rings, with black or white centres. Their action is analogous to that of thin crystallized plates, which either produce uniform tints like the laminæ of sulphate of lime, or uniaxal or biaxal systems of rings.

It would be unprofitable to describe minutely the great variety of phenomena which thin plates thus exhibit, as they vary with the refractive power of the fluid or solid upon which they are laid, so that I shall confine myself to the case in which a thin plate of oil of laurel rests on the surface of a specimen of *artificial realgar*. In common light, the colours of this film are very beautiful, but when examined in polarized light by an analyzing rhomb, they are brilliant beyond description.

1. *When the azimuth of the polarized light* is 90°, and the incidence of the polarized ray 56° 5′, the polarizing angle of oil of laurel.

When the film is viewed without the polarizing rhomb, no rings are seen, as there is no light reflected from the first surface of the film, and consequently no interference.

When the film is viewed with the polarizing rhomb, having its principal section in the plane of incidence, no rings appear, either in its ordinary or extraordinary image. But if the plane of polarization is less or more than 90°, by even a small quantity, then after the rhomb has been turned round nearly 90° towards the right, a system of *black-centred* rings is seen for an instant, and these, after disappearing, are followed by a system of *white-centred* ones, the white-centred rings appearing first if the rhomb is turned to the left. The same phenomena are repeated in every quadrant of the circular motion of the rhomb.

2. *When the azimuth of the polarized light varies* from 90° to 0°, the incidence, being 56° 5′, as before.

At 90° azimuth the phenomena are as above described.

* Philosophical Transactions, 1830, p. 147, fig. 3.

At $67\frac{1}{2}°$. Rhomb 0°, no rings.

Rhomb turning to the right, the white-centred rings appear, then vanish, when the azimuth of the rhomb is less than $67\frac{1}{2}°$; then black-centred rings appear, which vanish at 180°; then succeed the white-centred ones, which vanish at about 210°; then the black-centred, which continue to 360°.

At 45°, $22\frac{1}{2}°$. The very same phenomena appear at these and other azimuths, the azimuths of the rhomb at which the rings disappear out of the plane of incidence being a little less than the azimuths of the polarized light.

At 0°. The evanescence of the rings takes place when the azimuths of the rhomb are 0°, 90°, 180°, and 270°, the *white-centred* rings appearing in the *first* and *third*, and the *black-centred* ones in the *second* and *fourth* quadrant.

3. *Azimuth of polarized light* 90°.

Incidence of polarized light 68° 3′, the polarizing angle of realgar.

At this angle all the light reflected from the realgar has disappeared, excepting a dark bluish purple, in the middle of which is seen, without using the rhomb, a splendid system of richly-coloured rings, with a *white centre*. When the rhomb is applied as before, and performs a complete revolution, the white-centred rings are seen all round, disappearing at 90° and 270°.

4. *When the azimuth of the polarized light varies from* 90° to 0°, the incidence being 68° 3′, as before.

At 90° azimuth, the phenomena are as above described.

At 80°, and all other azimuths, the *white-centred rings* are seen when the rhomb is at 0°; but they disappear at azimuths of the rhomb a little less than the azimuths of polarization, and are then succeeded by the *black-centred rings*.

At 0° azimuth, the rings disappear when the rhomb is at 0° and 180°, and are *black-centred* all round.

Without using the rhomb, the rings always disappear at the azimuth x, at which the planes of polarization of the interfering pencils are rectangular.

At incidences above 68° 3′, the phenomena are of the same character. The rings are *white-centred* in 90° of azimuth, and when the rhomb is at 0°. They become very brilliant about 45°. Near 90° of rotation the rings vanish, and immediately the black-centred system appears, which quickly vanishes, and is succeeded by the white-centred system.

5. *Angles of incidence less than* 56° 5′.

In 90° of azimuth of the polarized ray, and the rhomb being at 0°, the black-centred rings are seen, and continue to be seen during a complete revolution of the rhomb. In all azimuths, from 90° to 0°, the rings disappear by turning the rhomb to the left, the arch diminishing from 90° to 0°; but in azimuths of an intermediate magnitude, the disappearance of the rings is followed by the appearance of the *white-centred system*, which quickly disappears, and is succeeded by the *black-centred system*. This phenomenon is seen best near 45° of azimuth.

When the plates or films are too thick to give the coloured rings, the phenomena of the differently polarized pencils may be finely seen by using *coloured glasses*, in which the pencils reflected from both surfaces may be observed. If the glass is *green*, for example, the pencil or image of a small aperture or luminous body will be *green*, while that reflected from the first surface, though in reality colourless, will appear *red*, from the physiological action of the green light upon the retina. Hence the two differently polarized pencils will have different colours, as if they were the tints of polarized light. If these coloured glasses are laid upon, or cemented on one side to, metals or highly refracting substances, the polarization of the coloured pencils which they reflect will be modified according to the principles already explained, and they will exhibit many interesting phenomena, varying with the colours of the glasses, as if the colours were produced by the absorption of polarized light.

In order to convey a general idea of the different classes of phenomena described in the preceding paper, I have represented two of the most important in figs. 4 and 5.

1. *Glass and Water.*—When a film of aqueous vapour is laid upon glass whose index of refraction is 1·508, the rings disappear at 53° 11′, the polarizing angle of the water, and also in the various azimuths where the two interfering pencils are polarized in planes at right angles to each other. At all azimuths greater than these, and at angles of incidence above the polarizing angle, the *white-centred* rings appear; and at all azimuths less than these, and at all incidences (except those at which the white-centred rings are seen), the *black*-centred rings appear.

The following Table shows the values of x, or the azimuths of disappearance of the rings, as computed from the formula in p. 49:—

Angles of Incidence.	Azimuths.	Complements.
° ′	° ′	° ′
53 11	90 0	0 0
55 0	82 8	7 52
60 0	76 52	13 8
65	75 15	14 45
67	75 10	14 50
70	75 30	14 30
73	76 18	13 42
74	76 42	13 18
75	77 9	12 51
76	77 36	12 24
80	80 0	10 0
85	84 15	5 45
90	90 0	0 0

If we now conceive A B, fig. 4, to be the section of the plane of incidence, having the different incidences marked upon it from 90° to 53° 11′, and if round a centre in A B prolonged, where 0° of incidence falls, we describe the azimuthal circle Z A Z,

then the complements of the azimuths of the polarized light being set off from the corresponding angles of incidence on each side of A B, the curves A C B, A C B passing through these points will show at what angles of incidence and azimuth the rings disappear, in consequence of the planes of polarization of the two pencils being at these places rectangular.

At all incidences, and in all azimuths within the shaded space A C B C, the *white-centred rings* are seen, and at all other azimuths and incidences the *black-centred* rings are seen.

2. *Fluor Spar and Water.*—I have taken this combination as a specimen of the phenomena which take place at some incidences less than 90°, when the refracted ray falls on the second surface of the film, at angles greater than its polarizing angle. The following Table shows the values of x and their complements:—

Angles of Incidence.	Azimuths.	Complements.
53° 11′	0° 0′	0° 0′
55	82 35	7 25
60	77 47	12 13
63	76 54	13 6
65	76 41	13 19
67	77 6	12 54
70	78 9	11 51
75	82 0	8 0
78	88 41	11 9
78 4	90 0	0 0
80	83 28	6 32
85	77 31	12 29
90	74 14	15 46

By projecting these values, as is done in fig. 5, we obtain a double set of curves which unite at D, where the angle of incidence is 78° 4′, at which the refracted ray falls upon the second surface at its polarizing angle.

At all incidences, and in all azimuths within the shaded portions of the figure Z A Z D, D C B C, the *white-centred rings* are seen. At all azimuths and incidences corresponding with the outlines of the curves Z D Z, D C B C, the *rings disappear*; and at all azimuths and incidences without the shaded portions of the figure, the *black-centred rings* are seen*.

* No reference is made in these figures to the phenomena which are seen by using both polarized light and the analyzing rhomb.

St. Leonard's, St. Andrew's,
April 8, 1841.

VI. *Memoir of the Case of a Gentleman born blind, and successfully operated upon in the 18th year of his age, with Physiological Observations and Experiments.* By J. C. August Franz, *of Leipzic, M.D., M.R.C.S., &c. Communicated by Sir* Benjamin C. Brodie, *Bart., F.R.S., &c.*

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Mr. F. J., the subject of the present memoir, is the son of a physician; of scrofulous diathesis, but otherwise of robust constitution; of irritable temperament, but of contented and happy disposition; and endowed with an excellent understanding, quick power of conception, and retentive memory. In both the eyes of his father, cataract (with the addition, I suspect, of glaucoma) has manifested itself within the last four years, after a severe attack of influenza. The relatives on the paternal side are predisposed to diseases of the eye, but in the mother, and in the relatives on her side, no such predisposition can be traced. With regard to the cause of the ophthalmic affections which form the subject of this paper, the mother seemed to lay much stress on the following circumstance, which, although it may possibly have had some share in the cause of one of them, can have had no influence, in my opinion, in producing the other. She stated to me that in the eighth month of her pregnancy, which up to this period had proceeded favourably, she received from her youngest child, which she was carrying in her arms, a severe blow on the eye. This accident caused inflammation of the eye, accompanied with a curious visual illusion, viz. that all objects which she saw, but especially those situated on the ground, appeared of a deep concave form; an illusion which lasted for several months. The fright experienced from the accident also brought on convulsions, which, recurring several times, extended even to the fœtus. The recurrence of these convulsions produced in the mind of the mother a continual anxiety and fear for the health of the child, while the pain arising from the ophthalmia, together with the visual illusion just mentioned, gave her fears a direction more especially towards its eyes. Delivery took place at the proper period, when the eyes of the infant, which was otherwise healthy and well-formed, were found to present a twofold defect of organization. The father, to whose statement, on account of his professional knowledge, more weight is to be attached, informed me that both eyes were turned inwards to such an extent that a portion of the cornea was hidden by the inner canthus, and that in both pupils a yellowish-white discoloration was to be observed, which, being situated behind the iris, could not be the pupillary membrane. That the strabismus and cataract of both eyes in this case were congenital, is evident from the

testimony both of the parents and of the nurse, whom I have closely questioned on this subject. The latter, who can distinctly remember all the circumstances of the case, told me that when the child was a few months old, she held a light before its eyes, of which it took no notice. I ascertained also from her that the eye-balls had not that restless motion which is generally observed in those who are born blind, but that both eyes were always turned inwards, and that but rarely either the one or the other was moved from the internal canthus.

It was also stated to me, that towards the end of the second year the operation of keratonyxis was performed on the right eye, upon which a severe iritis ensued, terminating in atrophy of the eye-ball. Within the next four years two similar operations were performed on the left eye, which did not indeed destroy the organ, but at the same time did not remove the opacity in the pupil. The colour of the opacity became in time, however, of a clearer white; and the patient acquired a certain sensation of light, which he did not seem to have had before the operation. Both eyes for a long time retained a disposition to inflammation, and suffered repeatedly from conjunctivitis, whence the vessels of the conjunctiva were increased in number and size to such an extent, that it was necessary they should be several times excised.

At the end of June 1840, the patient, being then seventeen years of age, was brought to me by my friend Dr. Swaine, for the purpose of consulting me with regard to the congenital double strabismus, and at the same time to hear my opinion on the more severe ophthalmic affection, which up to this period had been considered incurable; the patient himself regarded his case as hopeless. The following are the particulars elicited on an attentive and careful examination:—On the right side, the eyelids and parts adjacent appeared contracted; they were less in size, and the eye itself was situated deeper in the orbit than the left. At each act of winking spasms of the eyelids were induced, and, when the left eye was turned outwards, the spasmodic twitchings extended over that half of the face. Both eyes were so much inverted, that nearly one half of the cornea was hidden by the inner canthus. The left eye he could move voluntarily outwards or in any direction with certainty, but not without exertion; it returned immediately inwards, when the influence of the will had ceased. The motion of the right eye upwards and downwards the patient had under his control, but not so the movement towards the external canthus, in effecting which he only succeeded after many attempts. The left eye-ball was of the natural size and elasticity; the right, on the other hand, was at least a third smaller, and felt soft, and like dough; it was also, in the neighbourhood of the rectus internus, flat, or rather pressed inwards. The cornea was less convex, somewhat smaller, but not in proportion to the diminished size of the globe itself; it was clear, and free from opacity, except in the centre, where the keratonyxis had left an opake spot. The fibrous structure of the iris was irregular; its colour, which was brown, rather lighter than that of the left eye. Different degrees of light produced no effect on the motion of the iris; but when the eye was moved in a horizontal di-

rection outwards, the pupil, in passing the centre of the orbit, contracted a little, and, when approaching the outer canthus, expanded again to the size it held when the eye was in its usual inverted position. On looking from the temporal side into the pupil, a large portion of the opake capsule was observed in the posterior chamber. The interior of the eye-ball presented a brownish-black appearance. The patient had not the slightest perception of light with this eye; it was perfectly amaurotic. The left eye presented in the conjunctiva, especially at the inner canthus, a number of varicose vessels, and in the sclerotica a fine vascularity around the outer half of the cornea; this latter membrane was regularly convex, clear, and perfectly pellucid The fibres of the iris were rather irregular; its brown colour not equally diffused. The pupil, which was uncommonly large, was not round, but drawn angularly downwards and inwards, neither altering in dimension with the movements of the eye, nor from the stimulus of light. On examining the eye by looking straight into it through the pupil, the anterior wall of the capsule was observed undestroyed, rendered opake in its whole extent, hypertrophied in several places, and of a colour and lustre like mother-of-pearl. On looking from the temporal side in an oblique direction into the pupil, there was visible in the anterior wall of the capsule a very small perpendicular cleft of about one line and a quarter in length. This cleft was situated so far from the centre of the pupil that it was entirely covered by the iris, and the inferior border being united to the uvea, it was kept a little open, so that the aqueous humour had free entrance into the cavity of the capsule. Except at the spot where the union of the capsule with the uvea took place (the cause of the angular form of the pupil), these two membranes were not in contact with each other. The patient only complained of an occasional sensation of pressure in the interior of the eye; otherwise the organ was free from pain. With this eye he had a perception of light, and was even capable of perceiving colours of an intense and decided tone. He believed himself moreover able to perceive about one third of a square inch of any bright object, if held at the distance of half an inch or an inch from the eye, and obliquely in such a direction as to reflect the light strongly towards the pupil. But this I am convinced was a mere delusion; for, from the state of the interior of the eye, as just described, it is evident that all rays of light falling in the direction of the optic axis in the pupil must be intercepted, and reflected by the opake capsule. By these rays, therefore, a perception of light indeed might be conveyed, but certainly no perception of objects. On the other hand, it seems probable that the lateral cleft in the capsule permitted rays of light to pass into the interior of the eye. But as this small aperture was situated entirely behind the iris, those rays only would have permeated which came in a very oblique direction from the temporal side. Admitting then these rays of light to pass through the cleft, still on account of their obliquity they must fall at a place situated about midway between the ciliar ligament and the centre of the posterior hemisphere, where, from the laws of optics, they could produce but a very imperfect image; and, owing to this imperfection of the image, it was im-

possible that the portion of the retina upon which these rays impinged could have obtained that acuteness of sensation which is essential for the mental perception of the image caused by an object. Nevertheless, we will assume that the cleft in the capsule held the same relation to the eye in this instance, as a small hole in a card placed immediately before a healthy eye; in this case the patient would not only have seen an object at the distance of half an inch or an inch, but even at a much greater distance. That he was incapable of this I have satisfied myself by repeated experiments, which have led me to the conclusion that his belief that he really saw objects resulted solely from his imagination, combined with his power of reasoning. In feeling an object and bringing it in contact with the eyelids and the cheek, while holding it close before his eye, by his refined sense of touch an idea of the object was produced, which was judged of and corrected according to the experience he had gained by constant practice. This opinion is confirmed by the observations of those who have known and watched him for years, and also by a fact which I have myself frequently observed, viz. that all well-educated blind persons, who are not absolutely amaurotic, endeavour to persuade others that they see more than they really can, in order to conceal as much as possible their deficiency in the noblest of the senses, and from a reluctance to be regarded as objects of compassion.

On terminating this inquiry into the condition of the visual organ and the actual state of vision, I may here be allowed to mention that the patient's sense of touch had attained an extraordinary degree of perfection, and that in order to examine an object minutely he conveyed it to his lips. The sensation produced by silk stuffs was most pleasing to him. He was said to possess the power of distinguishing colours by the touch, but this assertion was not confirmed by his own testimony.

After the examination above detailed, I gave my opinion that the defect of the right eye was irremediable; that the patient might obtain sight with the left eye by an operation; and that the disfigurement caused by the inversion of the eyes might also be removed by operation. Though the left eye had been considered incurable like the right, there appeared to me reasonable grounds to hope for a cure, provided I could succeed in keeping down inflammation, which is not easy to be done in an eye already several times operated upon, and especially in a young plethoric subject. The operation was then resolved upon.

On the 10th of July 1840, in the presence of Dr. Swaine, and with the kind assistance of Messrs. F. Fowke and F. Steinhaeuser, I made an incision in the cornea upwards, and introducing a pair of fine curved forceps, armed with teeth, into the posterior chamber, I seized the anterior wall of the capsule by passing one of the blades of the forceps into its small aperture, and attempted by pulling it slowly to separate it from its adhesion with the uvea and its peripheral connexion, in which I succeeded without producing a prolapsus of the vitreous body, or tearing the capsule, which I now removed. After this proceeding, a large piece of the lens of an opake colour, probably the nucleus, presented itself in the pupil, which was easily removed

from the eye by means of DAVIEL's spoon; the pupillary aperture then appeared perfectly clear and black. The patient was now turned with his back to the light, for the purpose of trying a few experiments as to his sight, but from these I was obliged to desist on account of the pain which the light produced in the organ. Both eyes were then closed with narrow strips of court-plaster, and the patient carried to bed. Venesection, local bleeding, fomentations with iced water, continued without intermission for about forty-eight hours, together with the scrupulous observance of the most severe regimen, barely succeeded in keeping down the inflammation, the effects of which in this case, where but one eye offered hope, were much to be dreaded, if it should surpass that degree which was necessary for the healing of the wound in the cornea. This process went on and terminated so favourably, that the cicatrix, situated close to the sclerotica, is now scarcely visible. The patient suffered from muscæ volitantes and from a considerable intolerance of light, pain being produced by even a mild degree of light falling on the closed lids. The muscæ volitantes were greatly mitigated, and the intolerance of light ceased, after the lapse of a few weeks, by the use of proper pharmaceutical remedies, by local bleeding, change of air, &c., and the employment of the ophthalmic fountain of Professor JUNGKEN, which I have fully described in the Medical Gazette, vol. xxvii. p. 444. To promote the development of the power of vision, the use of the fountain was continued twice daily, with Pyrmont-water and latterly with simple spring-water, for the space of three months, when it was discontinued, as it began to irritate the eye.

Before I proceed further, I must again refer to the condition of sight previous to the operation. The right eye was completely amaurotic; in the left the power of vision existed, but, on account of the mechanical defect in the visual apparatus, was very little developed for the perception of light, and not at all for the perception of objects. It appeared to me, therefore, of the greatest interest to observe attentively the progressive development of the sensibility of the retina as regarded direct, refracted, reflected, and coloured rays of light; and also the progress of the visual perception in respect of the form, dimensions, and distance of objects. I was the more induced to undertake these physiological observations from having the opportunity of conducting them with an individual, who from his age, mental endowments, and education, offered peculiar advantages for such experiments.

On opening the eye for the first time on the third day after the operation, I asked the patient what he could see; he answered that he saw an extensive field of light, in which everything appeared dull, confused, and in motion. He could not distinguish objects. The pain produced by the light forced him to close the eye immediately. Two days afterwards, the eye, which had been kept closed by means of court-plaster, was again opened. He now described what he saw as a number of opake watery spheres, which moved with the movements of the eye, but, when the eye was at rest, remained stationary, and then partially covered each other. Two days after this the eye was again opened; the same phenomena were again observed, but the spheres were less opake and somewhat transparent; their movements more steady;

they appeared to cover each other more than before. He was now for the first time capable, as he said, to look through the spheres, and to perceive a difference, but merely a difference, in the surrounding objects. When he directed his eye steadily towards an object, the visual impression produced by the object was painful and very imperfect, and no clear visual perception of it took place, because the eye, on account of the intolerance of light, could not be kept open long enough for the formation of the idea as derived from visual sensation. The appearance of spheres diminished daily; they became smaller, clearer, and more pellucid, allowed objects to be seen more distinctly, and disappeared entirely after two weeks. The muscæ volitantes, which had the form of black, immoveable, and horizontal stripes, appeared, every time the eye was opened, in a direction upwards and inwards. When the eye was closed, he observed, especially in the evening, in an outward and upward direction, an appearance of dark blue, violet, and red colours; these colours became gradually less intense, were shaded into bright orange, yellow, and green, which latter colours alone eventually remained, and in the course of five weeks disappeared entirely.

As soon as the intolerance of light had so far abated that the patient could regard an object without pain and for a sufficient time to gain an idea of it, the following experiments were made in the presence of Dr. SWAINE. The first experiments were of that class in which the idea of a visible object is derived merely from pure visual sensation; the succeeding, of that kind in which the idea, in ordinary cases, depends upon the sense of sight combined with the sense of touch, and is gained by reflecting on the impressions made on the organs of both senses. It was necessary to perform these experiments on different days, as otherwise they would have distressed the eye too much.

1st Experiment. Silk ribands of different colours, fastened on a black ground, were employed to show, first the primitive, and then the complementary colours. The patient recognized the different colours, with the exception of yellow and green, which he frequently confounded, but could distinguish when both were exhibited at the same time. He could point out each colour correctly when a variety was shown him at the same time. Grey pleased him best, because this colour he said produced an agreeable and grateful sensation; the effect of red, orange, and yellow was painful, but not disagreeable; that of violet and brown not painful, but very disagreeable; the latter he called ugly. Black produced subjective colours, and white occasioned the recurrence of muscæ volitantes in a most vehement degree.

2nd Experiment. The patient sat with his back to the light, and kept his eye closed. A sheet of paper, on which two strong black lines had been drawn, the one horizontal, the other vertical, was placed before him, at the distance of about three feet. He was now allowed to open the eye, and, after attentive examination, he called the lines by their right denominations. When I asked him to point out with his finger the horizontal line, he moved his hand slowly, as if feeling, and pointed to the vertical, but after a short time, observing his error, he corrected himself. The outline in black of a square, six inches in diameter, within which a circle had been drawn, and within the latter a triangle, was, after careful examination, recognized and

correctly described by him. When he was asked to point out either of the figures, he never moved his hand directly and decidedly, but always as if feeling, and with the greatest caution; he pointed them out, however, correctly. A line consisting of angles, or in other words, a zigzag, and a spiral line, both drawn on a sheet of paper, he observed to be different, but could not describe them otherwise than by imitating their forms with his finger in the air. He said he had no idea of these figures.

3rd Experiment. The windows of the room were darkened, with the exception of one, towards which the patient, closing his eye, turned his back. At the distance of three feet and on a level with the eye, a solid *cube* and a *sphere*, each of four inches diameter, were placed before him. Allowing him to move the head in a lateral direction no further than was necessary to compensate the point of view of the right amaurotic eye, I now let him open his eye, and requested him to state decidedly what he observed. After attentively examining these bodies, he said he saw a *quadrangular* and a *circular* figure, and after some consideration he pronounced the one a *square* and the other a *disc*. His eye being then closed, the cube was taken away, and a disc of equal size substituted and placed next to the sphere. On again opening his eye, he observed no difference in these objects, but regarded them both as discs. The solid cube was now placed in a somewhat oblique position before the eye, and close beside it a figure cut out of pasteboard, representing a plane outline prospect of the cube when in this position. Both objects he took to be something like flat quadrates. A pyramid, placed before him with one of its sides towards his eye, he saw as a plain triangle. This object was now turned a little, so as to present two of its sides to view, but rather more of one side than of the other; after considering and examining it for a long time, he said that this was a very extraordinary figure; it was neither a triangle, nor a quadrangle, nor a circle; he had no idea of it, and could not describe it; "in fact," said he, "I must give it up." On the conclusion of these experiments, I asked him to describe the sensations the objects had produced, whereupon he said that immediately on opening his eye, he had discovered a difference in the two objects, the cube and the sphere, placed before him, and perceived that they were not drawings; but that he had not been able to form from them the idea of a square and a disc, until he perceived a sensation of what he saw in the points of his fingers, as if he really touched the objects. When I gave the three bodies (the sphere, cube, and pyramid) into his hand, he was much surprised that he had not recognized them as such by sight, as he was well acquainted with these solid mathematical figures by his touch. These experiments prove the correctness of the hypothesis I have advanced elsewhere on the well-known question put by Mr. Molyneux to Locke, which was answered by both these gentlemen in the negative, and has been much discussed since their time.

4th Experiment. In a vessel, containing water to about the depth of one foot, was placed a musket-ball, and on the surface of the water a piece of pasteboard, of the same form, size, and colour as the ball. The patient could perceive no difference in the position of these bodies; he believed both to be upon the surface of the water.

Pointing to the ball, I desired him to take up this object; he made an attempt to take it from the plane of the water, but when he found he could not grasp it there, he said he had deceived himself, the objects were lying in the water; upon which I informed him of their real position. I now desired him to touch the ball, which lay in the water, with a small rod; he attempted this several times, but always missed his aim; he could never touch the object at the first movement of his hand towards it, but only by feeling about with the rod. On being questioned with respect to reflected light, he said that he was always obliged to bear in mind, that the looking-glass was fastened to the wall, in order to correct his idea of the apparent situation of objects behind the glass.

When the patient first acquired the faculty of sight, all objects appeared to him so near that he was sometimes afraid of coming in contact with them, though they were in reality at a great distance from him. He saw everything much larger than he had supposed from the idea obtained by his sense of touch. Moving, and especially living objects, such as men, horses, &c., appeared to him very large. If he wished to form an estimate of the distance of objects from his own person, or of two objects from each other, without moving from his place, he examined the objects from different points of view by turning his head to the right and to the left. Of perspective in pictures he had of course no idea; he could distinguish the individual objects in a painting, but could not understand the meaning of the whole picture; it appeared to him unnatural, for instance, that the figure of a man represented in the front of the picture should be larger than a house or a mountain in the background. All objects appeared to him perfectly flat; thus, although he very well knew by his touch that the nose was prominent, and the eyes sunk deeper in the head, he saw the human face only as a plane. Though he possessed an excellent memory, this faculty was at first quite deficient as regarded visible objects; he was not able, for example, to recognize visitors, unless he heard them speak, till he had seen them very frequently. Even when he had seen an object repeatedly, he could form no idea of its visible qualities in his imagination, without having the real object before him. Heretofore, when he dreamed of any persons, of his parents, for instance, he felt them and heard their voices, but never saw them; but now, after having seen them frequently, he saw them also in his dreams. The human face pleased him more than any other object presented to his view; the eyes he thought most beautiful, especially when in motion; the nose disagreeable, on account of its form and great prominence; the movement of the lower jaw in eating he considered very ugly. Although the newly-acquired sense afforded him many pleasures, the great number of strange and extraordinary sights was often disagreeable and wearisome to him; he said that he saw too much novelty which he could not comprehend. And even though he could see both near and remote objects very well, he would nevertheless continually have recourse to the use of the sense of touch.

On the 21st of September I operated, in the presence of several medical gentlemen, in one sitting, on both eyes for the congenital strabismus. The lids were fixed by the

fingers of an assistant, the ball of the eye by a pair of forceps, and the tendon of the muscle divided by a pair of curved scissors. The rectus internus of the right eye was, like the organ itself, atrophied. The conjunctiva of the left eye was thickened at the inner angle; the muscle was uncommonly broad and thick; its tendon had a very broad attachment to the ball, and behind it was a separate bundle of muscular fibres attached to the sclerotica. The pupils of both eyes assumed immediately after the operation their proper position in the orbits. No inflammation ensued; not even in the left eye, which, from the prior operation, was still rather sensitive. The muscæ volitantes became less irksome, and the violent spasms which previously had affected not only the eyelids, but also the whole left side of the face, disappeared entirely. The right eye, which had been amaurotic, gained by this operation the power of perceiving light, so that when the left eye is closed, the patient can now distinguish light and shade, on the hand being moved before this eye. The sight of the left eye likewise was considerably improved in acuteness and clearness, both as regarded near and distant objects, but especially the latter. Objects now, however, appeared in a different situation to that which they really held; when, for instance, he directed his eye to an object situated immediately before him, he saw it more to his right, and, if he attempted to grasp it, he moved his hand in this wrong direction*. For this reason in walking across a room he always took a direction to the right, and consequently often came unawares in contact with articles of furniture, &c. This appearance of objects in false positions lasted for two months, after which time he was also capable of walking forwards in a straight direction. The right atrophied eye, which before the operation was deeply sunk in the orbit, is now more prominent, and appears therefore fuller and larger, so that the difference of the two eyes is less perceptible; he has consequently gained considerably in personal appearance. On one occasion when I was honoured with a visit from Mr. Lawrence, Dr. Watson, Dr. Kerrison, and several other medical gentlemen at my residence, I introduced him to them for examination.

In the middle of October I let him try several pair of spectacles at Mr. A. Ross's, in Regent Street. With a double convex lens of $5\frac{1}{2}$ inch focus, he saw both near and remote objects of large size most clearly and distinctly, but for small objects he could find no glasses that improved his sight. He recognized the capitals of a large print with his naked eye, and on looking through a pin-hole made in a card held close before the eye, he could distinguish even the small letters of a very minute print. He had not yet learned to read. The reason for the condition of his sight with respect to small objects, and that his vision is better on cloudy days, is no doubt to be sought in the enlarged pupil and the immobility of the iris.

In the middle of November he was able without spectacles to read the names over the windows of the shops in the streets, and to tell the time to the minute by St.

* This phenomenon I have observed in all eyes operated upon for strabismus of a great degree and long standing, when the other eye was closed. I have mentioned it in the Medical Gazette for June 1840, vol. xxvi. p. 540, where I have also given an explanation of the physiological cause.

Paul's clock. Walking alone in the crowded streets, especially in the City, he found very tedious. He said, seeing so many different things, and the quick movements of the multitude of people, carriages, &c., confused his sight to such a degree, that at last he could see nothing; that the sensation produced by the object last seen had not yet disappeared from the retina, when the next object made its impression thereon, by which means confusion of ideas, great anxiety, and even vertigo were occasioned, from which he could only free himself by closing his eyes for a few moments.

In the middle of December an experiment was again made with spectacles. A lens of seven inches focus was now of the same service as one of $5\frac{1}{4}$ inches had been two months before. After the operation for the strabismus he was accustomed, in speaking with any person, to turn his eye away from the face, as otherwise he said he felt disturbed by the looks of the person; he had now at length learned to look at the eyes of those with whom he conversed. The old habit of using the sense of touch to examine objects he had not yet entirely lost.

In the middle of February 1841, a third experiment was made with spectacles. A lens of ten inches focus was of the same service as one of seven inches had been on the last occasion, and one of $5\frac{1}{4}$ inches four months ago. This proves a slow, but positive amelioration of sight, and permits us to expect a still greater improvement, the more so as the patient has not passed the period of puberty. If the employment of spectacles were begun at the present period, although it is now more than seven months since the operation was performed, there would be no further amelioration of sight; the development of the visual apparatus would be arrested. I am therefore of opinion that the use of spectacles is not to be permitted, until it is, as it were, mathematically demonstrated by similar experiments with lenses, that the sight is no longer improved; by which means the faculty may in time, perhaps, reach such a degree of perfection as not to require any lens at all for remote objects.

This is the only case on record within my knowledge wherein, with a person born blind and afterwards successfully operated upon at a period of life as far advanced as in this instance, such experiments have ever been made. In the well-known case of Cheselden, published in the Philosophical Transactions for the year 1728 (page 447), the patient was only in the fourteenth year of his age, and although the case contains many highly interesting physiological observations, no series of systematic experiments was instituted. Beer has also made some interesting observations, which, however, like those made in rather a superficial manner by Janin and Daviel, tend principally to describe the impressions which the newly-acquired sense had made on the mind of the person operated upon. In Ware's case the patient was not born blind, but had become so at an early period of life. In the present paper I have merely given the simple history of the case, without making any remark on several points interesting to the pathologist and physiologist, to which I shall advert on a future occasion; the explanation and philosophy of the foregoing experiments as to the sense of sight I shall attempt in another paper, which I purpose to lay before this Society.

VII. *Additional Note on the Contraction of Voluntary Muscle in the Living Body.* By William Bowman, *Esq., F.R.S., Societ. Philomath. Paris. Corresp., Demonstrator of Anatomy in King's College, London, and Assistant Surgeon to the King's College Hospital.*

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In my paper of June last, published in the Philosophical Transactions*, I showed, by observations on the *Rigor Mortis*, that contraction, in voluntary muscle, essentially consists of an approximation and change of form of the minute particles composing its structure; the phenomena of contraction in living Monoculi and Arguli were also briefly adverted to, but it remained undecided in what manner these minute movements are employed in the higher animals, in the production of motion *during life*. The almost insurmountable difficulty of submitting the living muscle of the Vertebrata to high powers of the microscope, so much enhances the value of any facts bearing on this obscure point, that I am induced to lay before the Society a short account of some recent examinations of human tetanic muscle, which, with the considerations accompanying them, appear to me to afford conclusive evidence on the subject.

Two opportunities have lately occurred to me of carefully observing the conditions of the muscular system, in cases of fatal tetanus, and the following has been the result:

1. Many muscles appear healthy in all respects.

2. Parts of certain muscles present a remarkably pale gray aspect, arising, doubtless, from their blood having been pressed out by the contraction, a state of which the appearance has been aptly compared by my friend Professor Budd, to that of the flesh of fishes.

3. In other situations, the muscles have lost in a great measure their fine fibrous character, and present a soft mottled surface, which readily tears, or receives an impression from the contact of the finger, a condition with which may be associated,

4. Extensive ecchymoses, often contrasting strangely with the pallor of contiguous portions.

On microscopic examination, while the other affected muscles appear natural, the primitive fasciculi of those which have lost their texture or are ecchymosed (3. 4.), are by no means so, but present at certain points characteristic marks of a high degree of contraction; they are swollen into a fusiform shape, and have their transverse striæ very much closer together than usual (Plate II. (*a*)). Elsewhere these primitive fasciculi are, on the contrary, diminished in diameter, and their

* Part II., 1840.

transverse striæ either greatly widened and deranged (*b*), or altogether obliterated (*c*, *c*, *c*), in consequence of the whole texture of the organ being broken up into those primitive elements, of which the discs are constructed; and here the primitive fasciculi are frequently broken across, with or without a corresponding rupture of the sarcolemma (*d*, *e*). The extent of the swollen or contracted parts seems liable to great variety; the one selected for delineation (Pl. II.) contains upwards of sixty striæ, but others on contiguous primitive fasciculi were more extensive. Some primitive fasciculi in the neighbourhood, which at the point examined presented no rupture, had a very unusual diversity in the proximity of their striæ at different points, but everywhere preserved, like the rest, that proportion which I have shown to obtain between the diameter of the primitive fasciculi and the closeness of their transverse striæ. These I conclude to have been ruptured at a point further on.

Although the bare detail of these appearances may seem to warrant the conclusion that contractions have taken place in the situation of the fusiform or belly-like swellings, the effect of which has been to stretch and even to disorganize the remaining parts of the primitive fasciculi, yet I shall endeavour to confirm and illustrate it by the following considerations.

1. *The Contraction of a Muscle is the essential cause of its own rupture.*

This is best exemplified in a fragment of a primitive fasciculus of a reptile or fish removed from the body, and contracting between plates of glass. The contraction commences at its extremities, which, becoming swollen, receive the pressure of the upper plate, and may be fixed by it. If so, the intermediate part is stretched and torn as contraction proceeds; and if an isolated contraction occurs in the centre, the parts between it and the two extremities are similarly affected*, the conditions of the rupture being, 1. a partial contraction of the ruptured muscle, and, 2. a force superior to the tenacity of the uncontracted part, holding the ends of the fragment asunder.

The same conditions apply in the healthy living subject, where it is impossible, in consequence of the admirable adaptation of mechanical arrangements to the extensibility of muscles, that any rupture can take place solely from the action of antagonists. For example, no force of the flexors of the knee can by itself rupture the extensors, because the structure of the joint prevents flexion being carried beyond a point which the extensors, if relaxed, readily allow. And yet antagonist muscles may and do play a conspicuous part in most muscular ruptures; but it is only by affording a resistance to the approximation of the ends of the ruptured muscle, greater than the tenacity of its uncontracted parts,—such a resistance, in fact, as might be offered by any power mechanically adapted to produce the same effect. I say *uncontracted* parts, and the propriety of supposing them in the living subject appears from the examination of the tetanic muscle; for putting aside the impossibility of any rupture happening in a muscle of which no part was physically weaker than another,

* *Loc. cit.*, p. 490.

by a contraction consisting of an *absolutely simultaneous and uniform approximation* of all its elementary parts to one another, and supposing for a moment that though the contraction was everywhere present, yet some feebleness of structure, or trifling diminution of the contractile force at one part, determined a rupture to take place there, the resulting appearances would necessarily be altogether different from those that have been detailed. The rupture would be definite and abrupt, without extensive stretching and consequent disorganization, and the whole retracting fibre would bear traces of an uniform and universal contraction, instead of an unequal and partial one, very limited in extent, and similar in every essential character to that which I have delineated in dying muscle*.

2. *There is no Repellent Force between the Contractile Elements of Muscular Fibre.*

When muscle is taken from the body after all irritability has subsided, and when no stimulant has been previously applied capable of disordering its action during the rigor mortis, the distance between the transverse striæ will usually be found nearly uniform at every part; but when partial contractions occur in a fragment that has been removed prior to the cessation of its irritability, the contracted parts remain permanently distinguished from the rest of the primitive fasciculus, by the closeness of their transverse striæ. In the former case, every primitive fasciculus having its extremities held apart by its proper antagonists, the contractile efforts constituting the rigor mortis are uniformly and gradually expended, and no inordinate amount of contraction can leave its vestiges in any part; but, in the latter, no such antagonizing power being exerted, the contractions remain wherever and to whatever degree they may have been present.

This explanation involves and rests upon the above principle, and it follows, that whatever prevents a muscular fibre from being elongated, when its contractile energy subsides, must cause it to retain that arrangement of its parts which was assumed during contraction. Now, in the examples under consideration, the *rupture* prevented such an elongation, and the result is, that *the organ has been, as it were, surprised in the very act of contraction, and retains in its structure the permanent impression of that act*,—a view strongly corroborated by the uniformity of the distance between the transverse striæ in those muscles, which had been likewise convulsed, but had escaped rupture.

It may be further remarked, that the occurrence of ecchymoses entirely accords with the idea of partial contractions, while it is inconsistent with that of an universal one, for how could the vessels be torn in tetanic spasm if this were merely a strong and enduring, but uniform approximation of all the elements of the primitive fasciculi to one another? They would be compressed indeed, but not dragged asunder; whereas, such would be the natural effect of excessive partial contractions, oscillating from place to place, and continually drawing in opposite directions, and in an irregular manner, the uncontracted portions; for the capillaries take a longitudinal course be-

* *Loc. cit.*, Plate XIX. figs. 83. 88., No. 3. &c.

tween the primitive fasciculi, and inosculate with one another by very frequent transverse branches, which complete the vascular web, and serve to attach its several parts to the primitive fasciculi which occupy its interstices.

Lastly, all the specimens of bloodless tetanic muscle which I have seen, have presented this striking peculiarity:—that the pallor has not occupied the whole muscle, but patches of it, comprehending a portion of many primitive fasciculi, but not the entire length of any,—a fact tending to the same conclusion.

What has now been advanced, seems to render it—as it appears to me—certain, that the tetanic spasm has consisted in contractions engaging only parts of each primitive fasciculus at a time, and if so, of course changing their place, in order to bring every portion into use in its turn. Whether the primitive fasciculi alternate with one another in their contractions, is an obscure question, on which these observations shed no light.

It may be urged, however, that, even granting this conclusion true, it is unsafe to argue concerning the healthy and moderate actions of an organ from the phenomena it presents when in a morbid state. But the weight of this objection is more apparent than real, for in a physiological point of view, the contractions of tetanus differ from those properly termed voluntary, only in being uncontrollable by the will and excessive in amount and duration. In violent tetanic spasm, I have myself ascertained that the peculiar sound of voluntary contraction is audible in the part, and identical with the normal sound; and the appearance and feel of a muscle thus rigidly convulsed, can be perfectly simulated for a short period by an act of volition*.

I therefore conclude, from the whole of the preceding remarks, combined with the facts and arguments advanced in my former paper, that *the contraction of voluntary muscle is not a sustained act of the whole congeries of contractile elements composing it; but a rapid series of partial acts, in which all duly share, becoming by turns contracted and relaxed.*

King's College, London,
April 6, 1841.

The figure represents a portion of a primitive fasciculus, taken from among many others, from the complexus muscle, where it was ecchymosed and had lost in a great measure its fibrous appearance. At (*a*) a fusiform contracted portion, with the striæ remarkably close. On either hand the sarcous elements are much stretched (*b*), or even entirely disarranged (*c*, *c*, *c*), while at (*d*) there is a transverse rupture within the sarcolemma, and at (*e*) this sheath itself has given way.

* I yesterday, in company with my friend, Professor Todd, discovered the same appearances of *partial* contractions in the recti muscles of the abdomen, ruptured by violent straining in diarrhœa. The voluntary muscles of the whole body were infested with the *Trichina Spiralis*, and were enfeebled.—*August* 2, 1841.

VIII. *Note on an inequality in the Height of the Barometer, of which the Argument is the Declination of the Moon.* By Sir J. W. Lubbock, *Bart., Treas. and V.P.R.S.*

Received March 16,—Read March 18, 1841.

In the Companion to the British Almanac for 1839, I inserted some results which were obtained with a view of ascertaining the influence of the moon on the barometer and on the dew-point. Mr. Luke Howard's researches on this subject having recalled my attention to that paper, I find some results which I then gave seem to indicate that the moon's position in declination influences the barometer. In order to render this more manifest, I shall now combine all the observations given in p. 3*, (and here recapitulated) in three categories. These observations correspond in part to different angular distances of the moon from the sun (or times of transit); but as the inequality of the Ocean, of which the argument is the moon's declination, is independent (or very nearly so) of the time of the moon's transit, it is probable that so also is that in the height of the barometer. In this case we may with propriety combine in the same category observations which correspond to similar *declinations*, although to different times of transit.

The following are the results:

No. of Observations.	Moon's Declination.	Moon's Parallax.	Height of Barometer.	Thermometer Attached.
	°	′	inch.	
78	21·1	57·4	30·063	69·0
167	15·1	56·9	30·000	67·7
93	4·4	56·5	29·060	67·7

This seems to indicate an elevation of nearly one-tenth of an inch for seventeen degrees of declination. The inequality has a contrary sign to the inequality of the same argument in the Tides of the Ocean.

First Category.

No. of Observations.	Moon's Declination.	Moon's Parallax.	Height of Barometer.	Thermometer Attached.
	°	′	inch.	
12	21·7	58·3	30·022	69·7
13	21·6	58·5	30·091	69·9
13	20·2	56·8	30·057	67·2
16	21·8	56·4	30·106	67·6
12	21·2	56·7	30·037	69·4
12	20·3	58·0	30·068	70·5
78	126·8	344·7	180·381	414·3
Average..	21·1	57·4	30·063	69·0

* Companion to the British Almanac, 1839.

Second Category.

No. of Observations.	Moon's Declination.	Moon's Parallax.	Height of Barometer.	Thermometer Attached.
	°	′	inch.	
15	18·9	57·6	30·078	67·3
13	16·9	57·5	30·078	69·1
12	18·5	56·5	29·926	67·3
14	19·8	57·1	30·036	67·3
14	15·1	56·4	29·901	68·5
15	17·5	57·3	30·048	68·4
14	13·0	57·1	30·057	69·2
14	10·1	56·4	29·932	68·3
16	14·5	56·3	29·921	65·9
13	12·8	56·3	29·932	66·6
15	10·3	57·5	29·976	67·2
12	14·8	57·9	30·119	68·2
167	182·2	683·9	360·004	813·3
Average..	15·1	56·9	30·000	67·7

Third Category.

No. of Observations.	Moon's Declination.	Moon's Parallax.	Height of Barometer.	Thermometer Attached.
	°	′	inch.	
15	7·0	56·7	30·007	69·7
15	·2	56·4	30·009	69·3
16	4·3	56·4	29·986	68·4
17	9·0	56·4	29·880	66·3
14	1·7	56·2	29·941	65·9
16	4·2	56·9	29·939	66·6
93	26·4	339·0	179·762	406·2
Average..	4·4	56·5	29·960	67·7

IX. *On the Calculation of Attractions, and the Figure of the Earth.*

By C. J. Hargreave, *B.A., of University College, London. Communicated by* John T. Graves, *Esq., A.M., F.R.S., of the Inner Temple.*

Received January 14,—Read February 11, 1841.

THE principal object of the calculations contained in the following paper, is to investigate the figure which a fluid, consisting of portions varying in density according to any given law, would assume, when every particle is acted upon by the attraction of every other and by a centrifugal force arising from rotatory motion. To what extent this may have been the original condition of the earth, is a doubtful question; and although observation does not fully warrant this supposition of the regular arrangement of parts, it has necessarily been made the foundation of most of the mathematical calculations connected with the investigation. Before proceeding to this problem, it is necessary to calculate the attraction of a body of any given figure, and consisting of strata, varying in their densities according to any given law; and it is in this problem that the principal difficulty lies. The elegant method of solution discovered by Laplace is well known; and I have followed his steps as far as the point where the equation, known by his name, first appears. In order to illustrate the nature of the deviation which I have there made, it will be necessary to mention some of the principal steps of the two methods.

By means of a theorem, which Laplace laid down as true of all spheroids that differ but little from spheres, and the properties of the integral of the equation referred to, he was enabled to substitute the easy rules of differentiation for the more complicated inverse processes, and thus to compute the attraction of that class of figures. It has, however, been since discovered by Mr. Ivory, that this theorem is true only of spheroids of a particular kind; and, consequently, to this kind the solution of the problem is restricted. This defect, and the indirectness of his analysis, led other mathematicians to consider the question; and, in 1811, Mr. Ivory published his method, which has the great advantage of being more direct, though equally limited.

The method given in the following paper does not appear to be confined in its operation to any particular class of spheroids; since the coefficients of the series, into which the required function is developed, are determined absolutely, without any reference to the form of the spheroid to which they are about to be applied. The principal change consists in the different manner of treating this partial differential equation. Laplace and the subsequent writers on this equation, both as applied to the

calculation of attractions and the mathematical theory of electricity, suppose the coefficients of every term of the series to be expanded into another series of the sines and cosines of multiple arcs; and they avail themselves of the property which these terms possess of vanishing, in certain cases, when integrated between certain limits. The success of this plan, however, depends upon the restricting hypothesis above referred to, that the radius vector of the surface of the body is capable of expansion in a series of terms, each of which satisfies LAPLACE's equation. The following method shows that the coefficient of the general term of the first series is independent of one of the variables, and thus dispenses with the second series of expansions. This result I have arrived at, by first obtaining the integral of LAPLACE's equation in its most general form, and deducing the arbitrary functions introduced therein, from considerations which enter previous to the equation of the surface of the attracting body. These coefficients being known, it is evident that the attraction of any homogeneous body on a point within or without it may be immediately found when the equation of its surface is given, since it then depends only on a series of explicit and definite integrations of known functions, which can always be effected, at least approximately. From this, the attraction of a heterogeneous body, similarly circumstanced, may be found by the usual method of dividing it into concentric layers, and summing the several attractions of these, deduced as above.

By substituting the attraction so obtained, in the equation of equilibrium of a fluid body, CLAIRAUT's theorem is immediately deduced; and, from a peculiarity in the functions representing the attraction, it will be seen, that the same principles with longer processes may be carried on indefinitely, without the necessity of actually determining the precise form of those functions.

The restricted species of spheroid above referred to, comprises all surfaces of revolution; so that it is sufficiently extensive for most practical purposes; but the integration of LAPLACE's equation renders the analysis more direct, and the theory more complete.

On the General Problem of Attractions.

1. Let ϱ represent the density of a body at the point (x, y, z); and let f, g, h be the coordinates of a particle attracted by the body, parallel respectively to the axes x, y, z; then, if the power of attraction be inversely as the square of the distance, the resolved part of the attraction of the body, parallel to

$$x \text{ is } \iiint \frac{\varrho (f-x)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$y \text{ is } \iiint \frac{\varrho (g-y)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$z \text{ is } \iiint \frac{\varrho (h-z)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

the limits of integration being determined by the equation to the surface of the body.

2. Let V represent the sum of the products of each particle by the reciprocal of its distance from the attracted point.

Then $V = \iiint \frac{\varrho\, dx\, dy\, dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{1}{2}}}$; and, by differentiating V, we obtain the well-known property $\frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = 0$, or $-4\pi\varrho'$, according as the attracted particle is not or is within the attracting mass; ϱ' being in the latter case the density of the attracted particle*. By transforming these equations to polar coordinates, we obtain

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} + \frac{1}{r^2}\frac{d^2V}{d\theta^2} + r^2\cot\theta\frac{dV}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\varphi^2} = 0, \text{ or } -4\pi\varrho',$$

and

$$V = \int_0^r\int_0^\pi\int_0^{2\pi} \frac{\varrho\, r'^2 dr' \sin\theta' d\theta' d\varphi'}{\{r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'))\}^{\frac{1}{2}}};$$

where $r^2 = f^2 + g^2 + h^2$, $\cos\theta = \frac{h}{\sqrt{(f^2+g^2+h^2)}}$, $\tan\varphi = \frac{g}{f}$; and similar expressions are true of r' θ' and φ' in terms of x, y, z.

Put $\cos\theta = \mu$, and $\cos\theta' = \mu'$, and they become

$$r\frac{d^2(rV)}{dr^2} + \frac{d}{d\mu}\left((1-\mu^2)\frac{dV}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{d^2V}{d\varphi^2} = 0, \text{ or } -4\pi\varrho' r^2\dagger. \quad . \quad . \quad . \quad . \quad (1.)$$

$$V = \int_0^r\int_{-1}^1\int_0^{2\pi} \frac{\varrho\, r'^2 dr'\, d\mu'\, d\varphi'}{\{r^2 + r'^2 - 2rr'(\mu\mu' + \sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\varphi - \varphi'))\}^{\frac{1}{2}}} \quad . \quad (2.)$$

3. Expansion by the binomial theorem shows that

$$\{r^2 + r'^2 - 2rr'(\mu\mu' + \sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\varphi - \varphi'))\}^{\frac{1}{2}}$$

may be expressed either in powers of r or of r'; thus

$$P_0\frac{1}{r'} + P_1\frac{r}{r'^2} + \ldots + P_n\frac{r^n}{r'^{n+1}} + \ldots, \text{ or } P_0\frac{1}{r} + P_1\frac{r'}{r^2} + P_2\frac{r'^2}{r^3} + \ldots + P_n\frac{r'^n}{r^{n+1}} + \ldots,$$

where P_n is a symmetrical function of μ, $\sqrt{(1-\mu^2)}\cos\varphi$, $\sqrt{(1-\mu^2)}\sin\varphi$ on the one part, and μ', $\sqrt{(1-\mu'^2)}\cos\varphi'$, $\sqrt{(1-\mu'^2)}\sin\varphi'$ on the other.

By substituting the first expansion in (2.), and the value of V so obtained in (1.), we have a series of equations

$$\int_0^r\int_{-1}^1\int_0^{2\pi} \varrho\left\{\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_n}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{d^2P_n}{d\varphi^2} + n(n+1)P_n\right\}\frac{dr'\, d\mu'\, d\varphi'}{r'^{n-1}}$$
$$= 0, \text{ or } -4\pi\varrho',$$

except when $n = 2$; and in all cases

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_n}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{d^2P_n}{d\varphi^2} + n(n+1)P_n = 0; \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

which is the equation of LAPLACE's coefficients.

* Vide Pratt. Mec. Phil., § 168. LAPLACE, Méc. Cél. liv. iii. † Vide Pratt. Mec. Phil., § 169.

4. This equation was not integrated; but by a skilful use of its properties, the problem of attractions was greatly simplified by LAPLACE. He laid down a theorem, respecting the surfaces of all spheroids of small deviation, that their radii vectores might be developed into series, every term of which would satisfy the above equation; and he also gave a method of expansion. By means of this theorem, the problem could be solved for spheroidal bodies which differ but little from spheres; but its generality has been greatly restricted by the researches of subsequent writers*, by whom it has been shown that it is true only for bodies whose radii are expressible in *rational and integral functions of* μ', $\sqrt{(1-\mu'^2)}\cos\varphi'$, $\sqrt{(1-\mu'^2)}\sin\varphi'$. Among these are the ellipsoid and elliptical spheroid, and a large class of other spheroids. In these papers I have adopted a different proceeding; I integrate the equation itself generally, and determine the arbitrary functions contained in the integrals by the circumstances of the problem itself. In consequence of the peculiar form which P_n then takes, V may be found by effecting the operations indicated, which are only explicit integrations.

5. I shall now proceed to integrate this equation.

Consider μ and φ as functions of two new variables X and Y, to be determined from the equations,

$$dX = \frac{dX}{d\varphi}(d\varphi + k\,d\mu)$$

$$dY = \frac{dY}{d\varphi}(d\varphi + k'\,d\mu),$$

where k and k' are the roots of the equation $(1-\mu^2)k^2 + \frac{1}{1-\mu^2} = 0$. These roots are $\frac{\pm\sqrt{-1}}{1-\mu^2}$, whence we obtain

$$X = \varphi + \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}, \text{ and } Y = \varphi - \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}. \quad . \quad . \quad (4.)$$

$$\frac{d^2P_n}{d\mu^2} = \left(\frac{dX}{d\mu}\right)^2\frac{d^2P_n}{dX^2} + 2\frac{dX}{d\mu}\frac{dY}{d\mu}\frac{d^2P_n}{dX\,dY} + \left(\frac{dY}{d\mu}\right)^2\frac{d^2P_n}{dY^2} + \frac{d^2X}{d\mu^2}\frac{dP_n}{dX} + \frac{d^2Y}{d\mu^2}\frac{dP_n}{dY},$$

$$= -\frac{1}{(1-\mu^2)^2}\frac{d^2P_n}{dX^2} + \frac{2}{(1-\mu^2)^2}\frac{d^2P_n}{dX\,dY} - \frac{1}{(1-\mu^2)^2}\frac{d^2P_n}{dY^2} + \frac{2\mu\sqrt{-1}}{(1-\mu^2)^2}\left(\frac{dP_n}{dX} - \frac{dP_n}{dY}\right);$$

$$\frac{dP_n}{d\mu} = \frac{dX}{d\mu}\frac{dP_n}{dX} + \frac{dY}{d\mu}\frac{dP_n}{dY} = \frac{\sqrt{-1}}{1-\mu^2}\left(\frac{dP_n}{dX} - \frac{dP_n}{dY}\right);$$

$$\frac{d^2P_n}{d\varphi^2} = \left(\frac{dX}{d\varphi}\right)^2\frac{d^2P_n}{dX^2} + 2\frac{dX}{d\varphi}\frac{dY}{d\varphi}\frac{d^2P_n}{dX\,dY} + \left(\frac{dY}{d\varphi}\right)^2\frac{d^2P_n}{dY^2} + \frac{d^2X}{d\varphi^2}\frac{dP_n}{dX} + \frac{d^2Y}{d\varphi^2}\frac{dP_n}{dY},$$

$$= \frac{d^2P_n}{dX^2} + 2\frac{d^2P_n}{dX\,dY} + \frac{d^2P_n}{dY^2}.$$

Substituting these in (3.), we obtain

$$\frac{4}{1-\mu^2}\frac{d^2P_n}{dX\,dY} + n(n+1)P_n = 0.$$

* See two articles by Mr. IVORY in the Philosophical Transactions, 1812.

From (4.), by subtraction,

$$X - Y = \sqrt{-1} \log \frac{1+\mu}{1-\mu};$$

whence

$$\mu = \frac{\epsilon^{-(X-Y)\sqrt{-1}} - 1}{\epsilon^{-(X-Y)\sqrt{-1}} + 1},$$

and

$$1 - \mu^2 = \frac{4\,\epsilon^{-(X-Y)\sqrt{-1}}}{\left(\epsilon^{-(X-Y)\sqrt{-1}} + 1\right)^2} = \frac{4}{\left(\epsilon^{-\frac{1}{2}(X-Y)\sqrt{-1}} + \epsilon^{\frac{1}{2}(X-Y)\sqrt{-1}}\right)^2} = \frac{1}{\cos^2 \frac{X-Y}{2}};$$

consequently

$$\frac{d^2 P_n}{dX\,dY} + \frac{n(n+1) P_n}{4 \cos^2 \frac{X-Y}{2}} = 0. \quad \text{Let } n(n+1) = a.$$

Let

$$\frac{dP_n}{dY} = v, \text{ then } \frac{dv}{dX} + \frac{a P_n}{4 \cos^2 \frac{X-Y}{2}} = 0, \text{ and } P_n = -\frac{dv}{dX} \frac{4}{a} \cos^2 \frac{X-Y}{2};$$

$$v = \frac{dP_n}{dY} = -\frac{d^2 v}{dX\,dY} \frac{4}{a} \cos^2 \frac{X-Y}{2} - \frac{dv}{dX} \frac{4}{a} \cos \frac{X-Y}{2} \sin \frac{X-Y}{2},$$

or

$$\frac{d^2 v}{dX\,dY} + \frac{dv}{dX} \tan \frac{X-Y}{2} + \frac{a v}{4 \cos^2 \frac{X-Y}{2}} = 0.$$

Let

$$\frac{dv}{dY} + v \tan \frac{X-Y}{2} = t.$$

Then

$$\frac{d^2 v}{dX\,dY} + \frac{dv}{dX} \tan \frac{X-Y}{2} + \frac{v}{2 \cos^2 \frac{X-Y}{2}} = \frac{dt}{dX},$$

and

$$\frac{dt}{dX} + \frac{a-2}{4} \frac{v}{\cos^2 \frac{X-Y}{2}} = 0; \text{ or } v = -\frac{4}{a-2} \cos^2 \frac{X-Y}{2} \frac{dt}{dX};$$

whence

$$\frac{dv}{dY} = -\frac{4}{a-2} \cos^2 \frac{X-Y}{2} \frac{d^2 t}{dX\,dY} - \frac{4}{a-2} \cos \frac{X-Y}{2} \sin \frac{X-Y}{2} \frac{dt}{dX};$$

$$t = -\frac{4}{a-2} \cos^2 \frac{X-Y}{2} \frac{d^2 t}{dX\,dY} - \frac{8}{a-2} \cos \frac{X-Y}{2} \sin \frac{X-Y}{2} \frac{dt}{dX},$$

or

$$\frac{d^2 t}{dX\,dY} + \frac{dt}{dX} 2 \tan \frac{X-Y}{2} + t \frac{a-2}{4 \cos^2 \frac{X-Y}{2}} = 0.$$

Let

$$\frac{dt}{dY} + 2t \tan \frac{X-Y}{2} = q,$$

and by repeating a similar process, we obtain

$$\frac{d^2q}{dX\,dY}+\frac{dq}{dX}3\tan\frac{X-Y}{2}+q\,\frac{a-6}{4\cos^2\frac{X-Y}{2}}=0.$$

By observing the assumptions here made, and the results obtained, we find that in the first assumption $\left(\frac{dP_0}{dY}=v\right)$, the coefficient of $P_n\tan\frac{X-Y}{2}$ is 0; in the second, that of $v\tan\frac{X-Y}{2}=1$; and so on, in the order of the natural numbers; and in the results, the numerical coefficients are $1,\frac{a}{4};\ 2,\frac{a-2}{4};\ 3,\frac{a-6}{4}\ldots$ generally $n,\frac{a-n(n-1)}{4}$. I shall prove this in the general case, by showing that if it is true of one value of n (as we see it is), it is true of the next value, and so on. Let the $(n-1)$th substitution give

$$\frac{d^2\varrho}{dX\,dY}+\frac{d\varrho}{dX}(n-1)\tan\frac{X-Y}{2}+\varrho\frac{a-(n-1)(n-2)}{4\cos^2\frac{X-Y}{2}}=0,$$

and let

$$\frac{d\varrho}{dY}+(n-1)\,\varrho\tan\frac{X-Y}{2}=s:$$

then, as before,

$$\frac{d^2\varrho}{dX\,dY}+(n-1)\tan\frac{X-Y}{2}\cdot\frac{d\varrho}{dX}+\frac{n-1}{2}\,\frac{1}{\cos^2\frac{X-Y}{2}}\cdot\varrho=\frac{ds}{dX};$$

and, therefore,

$$\frac{ds}{dX}+\varrho\frac{a-(n-1)n}{4\cos^2\frac{X-Y}{2}}=0,\text{ and }\varrho=-\frac{ds}{dX}\frac{4}{a-(n-1)n}\cos^2\frac{X-Y}{2};$$

$$\frac{d\varrho}{dY}=\frac{d^2s}{dX\,dY}\frac{4}{a-(n-1)n}\cos^2\frac{X-Y}{2}-\frac{ds}{dX}\frac{4\cos\frac{X-Y}{2}\sin\frac{X-Y}{2}}{a-(n-1)n}.$$

Consequently

$$-\frac{d^2s}{dX\,dY}\frac{4}{a-(n-1)n}\cos^2\frac{X-Y}{2}-\frac{ds}{dX}\frac{4}{a-(n-1)n}\cos\frac{X-Y}{2}\sin\frac{X-Y}{2}$$
$$-\frac{ds}{dX}\frac{4(n-1)}{a-(n-1)n}\cos\frac{X-Y}{2}\sin\frac{X-Y}{2}=s,$$

or

$$\frac{d^2s}{dX\,dY}\frac{4}{a-(n-1)n}\cos^2\frac{X-Y}{2}+\frac{ds}{dX}\frac{4n}{a-(n-1)n}\cos\frac{X-Y}{2}\sin\frac{X-Y}{2}+s=0;$$

that is,

$$\frac{d^2s}{dX\,dY}+\frac{ds}{dX}n\tan\frac{X-Y}{2}+s\frac{a-n(n-1)}{4}\cos^2\frac{X-Y}{2}=0,$$

and, therefore, the law of coefficients, as above stated, is correct.

Restoring the value of a, we get

$$\frac{d^2 s}{dX\,dY} + \frac{ds}{dX} n \tan\frac{X-Y}{2} + \frac{s.n}{2}\frac{1}{\cos^2\frac{X-Y}{2}} = 0,$$

which is integrable; and its integral is $\frac{ds}{dY} + n \,.\, s \tan\frac{X-Y}{2} =$ some arbitrary function of Y, as χ Y. Integrate again, then

$$s = \varepsilon^{-\int n \tan\frac{X-Y}{2} dY}\left(\int \varepsilon^{\int n \tan\frac{X-Y}{2} dY} \chi Y\, dY + \psi X\right),$$

where ψ is arbitrary. Effecting these integrations, and reducing,

$$s = \cos^{-2n}\frac{X-Y}{2}\left(\int \cos^{2n}\frac{X-Y}{2} \,.\, \chi Y\, dY + \psi X\right).$$

To return to P_n, we have the following systems of equations:

$P_n = \int v\, dY,$

$$v = \varepsilon^{-\int \tan\frac{X-Y}{2} dY}\left(\int t\varepsilon^{\int \tan\frac{X-Y}{2} dY} dY\right) = \cos^{-2}\frac{Y-X}{2}\int t \cos^2\frac{Y-X}{2} dY,$$

$$t = \cos^{-4}\frac{Y-X}{2}\int q \cos^4\frac{Y-X}{2} \,.\, dY,$$

$$\vdots$$

$$\varrho = \cos^{-2(n-1)}\frac{Y-X}{2}\int s \cos^{2(n-1)}\frac{Y-X}{2} \,.\, dY,$$

$$s = \cos^{-2n}\frac{Y-X}{2}\left(\int \cos^{2n}\frac{Y-X}{2} \,.\, \chi Y\, dY + \psi X\right);$$

whence

$$P_n = \ldots\ldots \int \cos^{-2}\frac{Y-X}{2}\int \cos^{-2}\frac{Y-X}{2}\left(\int \cos^{2n}\frac{X-Y}{2}\, \chi Y\, dY + \psi X\right) dY\, dY \ldots (n \text{ times.})$$

Now $\cos\frac{Y-X}{2} = \cos\left(\sqrt{-1}\,\log\sqrt{\frac{1+\mu}{1-\mu}}\right) = \frac{1}{2}\left(\sqrt{\frac{1+\mu}{1-\mu}} + \sqrt{\frac{1-\mu}{1+\mu}}\right),$

and $\cos^2\frac{Y-X}{2} = \frac{1}{1-\mu^2}$; and the complete integral will be expressed, by substituting for X and Y in terms of μ and φ.

6. But an important point yet remains to be determined. The original equation, being a partial differential equation of the second order, can only involve in its integral two arbitrary functions. But here, after χ Y and ψ X have come in by two integrations, we have n integrations to perform with respect to Y. It would seem, therefore, that no constant or arbitrary function of X must be added in these integrations.

Such is not the case. At each integration a function of X must be added, and these functions determined by reference to the original differential equation*.

7. Returning to the value of P_n, we have

$$P_0 = \chi_1\left(\varphi - \tfrac{1}{2}\sqrt{-1}\log\tfrac{1+\mu}{1-\mu}\right) + \psi\left(\varphi + \tfrac{1}{2}\sqrt{-1}\log\tfrac{1+\mu}{1-\mu}\right).$$

Now in the calculation of attractions, where P_0 is the coefficient of r^0 in the expansion of

$$\left\{r^2 + r'^2 - rr'\left(\mu\mu' + \sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\varphi-\varphi')\right)\right\}^{-\frac{1}{2}},$$

we know that it is 1; consequently

$$\chi_1\left(\varphi - \tfrac{1}{2}\sqrt{-1}\log\tfrac{1+\mu}{1-\mu}\right) + \psi\left(\varphi + \tfrac{1}{2}\sqrt{-1}\log\tfrac{1+\mu}{1-\mu}\right) = 1,$$

and expanding by Taylor's theorem, we get

$$\left.\begin{aligned} &\chi_1\varphi - \chi_1'\varphi\,\tfrac{1}{2}\sqrt{-1}\log\tfrac{1+\mu}{1-\mu} - \frac{\chi_1''\varphi}{2}\left(\tfrac{1}{2}\log\tfrac{1+\mu}{1-\mu}\right)^2 \\ &\quad + \frac{\chi_1'''\phi}{2.3}\sqrt{-1}\left(\tfrac{1}{2}\log\tfrac{1+\mu}{1-\mu}\right)^3 + \&c. \\ &+\psi\varphi + \psi'\varphi\,\tfrac{1}{2}\sqrt{-1}\log\tfrac{1+\mu}{1-\mu} - \frac{\psi''\phi}{2}\left(\tfrac{1}{2}\log\tfrac{1+\mu}{1-\mu}\right)^2 \\ &\quad - \frac{\psi'''\phi}{2.3}\sqrt{-1}\left(\tfrac{1}{2}\log\tfrac{1+\mu}{1-\mu}\right)^3 + \&c. \end{aligned}\right\} = 1.$$

By equating the coefficients of the same powers of $\frac{1}{2}\log\frac{1+\mu}{1-\mu}$, we have $\chi_1\varphi + \psi\varphi = 1$, and $\psi'\varphi - \chi_1'\varphi = 0$, or $\psi\varphi - \chi_1\varphi =$ constant.

Therefore $\psi\varphi$ and $\chi_1\varphi$ are absolute constants, and their sum is 1; whence it follows that $\chi\varphi = 0$. Let $\psi\phi = \frac{c}{2}$, then

$$P_n = c\left(\ldots\int\cos^{-2}\tfrac{Y-X}{2}\int\cos^{-2}\tfrac{Y-X}{2}\int\tfrac{1}{2}\cos^{-2}\tfrac{Y-X}{2}\,dY\,dY\ldots(n\text{ times})\right).$$

Effecting these integrations, we find that P_n consists of a series of powers of

* The common differential equation $(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + n(n+1)P_n = 0$ will illustrate this point. Let $P_n = \frac{d^{-n}z}{d\mu^{-n}}$, and after substitution, differentiate n times; then $(1-\mu^2)\frac{d^2z}{d\mu^2} - 2(n+1)\mu\frac{dz}{d\mu} = 0$, whence $z = \int\frac{k\,d\mu}{(1-\mu^2)^{n+1}} + m$. It is clear that no more *arbitrary* constants than k and m can be introduced; and yet if the integrals were left indefinite, we might obtain an integral of an expression which should differ from the integral of the same expression obtained by a slightly different process, by a constant. By another integration this would cease to be a constant, and we should obtain thus different values for P_n. The fact is, that constants must be added at each integration, and recourse had to the original equation, to determine them in terms of m, k, and μ.

$\tan\frac{Y-X}{2}$, whose coefficients may, for anything we yet know, be functions of X. The following process shows that constants as coefficients will satisfy the original equation, and determines them. The integration itself gives the coefficient of the highest power.

Let then*

$$P_n = c\left\{\begin{array}{l}\frac{2^n-1}{[n]}\tan^n\frac{Y-X}{2} + c'\tan^{n-1}\frac{Y-X}{2} + c''\tan^{n-2}\frac{Y-X}{2} \\ + \ldots + c^{(n-2)}\tan^2\frac{Y-X}{2} + c^{(n-1)}\tan\frac{Y-X}{2} + c^{(n)}\end{array}\right\}.$$

Then

$$\frac{dP_n}{dY} = c\left\{\begin{array}{l}\frac{2^{n-2}}{[n-1]}\tan^{n-1}\frac{Y-X}{2} + \frac{n-1}{2}c'\tan^{n-2}\frac{Y-X}{2} \\ + \frac{n-2}{2}c''\tan^{n-3}\frac{Y-X}{2} + \ldots + \frac{2}{2}c^{(n-2)}\tan\frac{Y-X}{2} \\ + \frac{1}{2}c^{(n-1)}\end{array}\right\}\left(1+\tan^2\frac{Y-X}{2}\right),$$

$$= c\left\{\begin{array}{l}\frac{2^{n-2}}{[n-1]}\tan^{n+1}\frac{Y-X}{2} + \frac{n-1}{2}c'\tan^n\frac{Y-X}{2} + \left(\frac{n-2}{2}c'' + \frac{2^{n-2}}{[n-1]}\right)\tan^{n-1}\frac{Y-X}{2} \\ + \left(\frac{n-3}{2}c''' + \frac{n-1}{2}c'\right)\tan^{n-2}\frac{Y-X}{2} \\ + \left(\frac{n-4}{2}c^{iv} + \frac{n-2}{2}c''\right)\tan^{n-3}\frac{Y-X}{2} + \ldots + \left(\frac{3}{2}c^{(n-3)} + \frac{5}{2}c^{(n-5)}\right)\tan^4\frac{Y-X}{2} \\ + \left(\frac{2}{2}c^{(n-2)} + \frac{4}{2}c^{(n-4)}\right)\tan^3\frac{Y-X}{2} \\ + \left(\frac{1}{2}c^{(n-1)} + \frac{3}{2}c^{(n-3)}\right)\tan^2\frac{Y-X}{2} + \frac{2}{2}c^{(n-2)}\tan\frac{Y-X}{2} + \frac{1}{2}c^{(n-1)}\end{array}\right\}.$$

$$\frac{d^2P_n}{dX\,dY} = -\frac{c}{\cos^2\frac{Y-X}{2}}\left\{\begin{array}{l}\frac{(n+1)2^{n-3}}{[n-1]}\tan^n\frac{Y-X}{2} + \frac{n(n+1)}{4}c'\tan^{n-1}\frac{Y-X}{2} \\ + \frac{n-1}{2}\left(\frac{n-2}{2}c'' + \frac{2^{n-2}}{[n-1]}\right)\tan^{n-2}\frac{Y-X}{2} \\ + \frac{n-2}{2}\left(\frac{n-3}{2}c''' + \frac{n-1}{2}c'\right)\tan^{n-3}\frac{Y-X}{2} \\ + \ldots + \frac{4}{2}\left(\frac{3}{2}c^{(n-3)} + \frac{5}{2}c^{(n-5)}\right)\tan^3\frac{Y-X}{2} \\ + \frac{3}{2}\left(\frac{2}{2}c^{(n-2)} + \frac{4}{2}c^{(n-4)}\right)\tan^2\frac{Y-X}{2} \\ + \frac{2}{2}\left(\frac{1}{2}c^{(n-1)} + \frac{3}{2}c^{(n-3)}\right)\tan\frac{Y-X}{2} - \frac{1}{2}\frac{2}{2}c^{(n-2)}\end{array}\right\}.$$

* $[n] = 1.2.3.4\ldots.n.$

$$n\frac{n+1}{4}\,P_n\frac{1}{\cos^2\frac{Y-X}{2}}=\frac{c}{\cos^2\frac{Y-X}{2}}\left\{\begin{array}{l}\frac{(n+1)2^{n-3}}{[n-1]}\tan^n\frac{Y-X}{2}+\frac{n(n+1)}{4}c'\tan^{n-1}\frac{Y-X}{2}\\ +\frac{n(n+1)}{4}c''\tan^{n-2}\frac{Y-X}{2}+\ldots.\\ \ldots+\frac{n(n+1)}{4}c^{(n-1)}\tan\frac{Y-X}{2}+\frac{n(n+1)}{4}c^{(n)}\end{array}\right\}.$$

Whence we obtain $c'=0$, $c'''=0$, $c^{v}=0$, &c.

Also

$$\left(\frac{n(n+1)}{4}-\frac{(n-1)(n-2)}{4}\right)c''=\frac{2^{n-3}}{[n-2]};\ \left(\frac{n(n+1)}{4}-\frac{(n-3)(n-4)}{4}\right)c^{iv}=\frac{(n-3)(n-2)}{4}c'';$$

$$\left(\frac{n(n+1)}{4}-\frac{(n-5)(n-6)}{4}\right)c^{vi}=\frac{(n-5)(n-4)}{4}c^{iv};\ \&c.\ \&c.$$

Consequently

$$P_n=c\left\{\begin{array}{l}\frac{2^{n-1}}{[n]}\tan^n\frac{Y-X}{2}+\frac{2^{n-1}}{2[n-2](2n-1)}\tan^{n-2}\frac{Y-X}{2}\\ +\frac{2^{n-1}}{2.4.[n-4](2n-1)(2n-3)}\tan^{n-4}\frac{Y-X}{2}\\ +\frac{2^{n-1}\tan^{n-6}\frac{Y-X}{2}}{2.4.6.[n-6].(2n-1)(2n-3).(2n-5)}+..\end{array}\right\}.$$

Now

$$\frac{Y-X}{2}=-\frac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}=\tan^{-1}\left(-\mu\sqrt{-1}\right),\ldots\tan\frac{Y-X}{2}$$

$$=\left(-\mu\sqrt{-1}\right);$$

whence, finally,

$$P_n=2^{n-1}c\left(\frac{(-\mu\sqrt{-1})^n}{[n]}+\frac{(-\mu\sqrt{-1})^{n-2}}{2.[n-2].(2n-1)}+\frac{(-\mu\sqrt{-1})^{n-4}}{2.4.[n-4](2n-1)(2n-3)}+\ldots\right),$$

a remarkable result, showing that in this instance P_n is independent of φ.

P_n being free from φ, is a perfectly symmetrical function of μ and μ'; and μ' is a constant with respect to μ; therefore

$$P_n=K_n\left(\frac{(-\mu\sqrt{-1})^n}{[n]}+\frac{(-\mu\sqrt{-1})^{n-2}}{2[n-2](2n-1)}+\frac{(-\mu\sqrt{-1})^{n-4}}{2.4.[n-4](2n-1)(2n-3)}+\ldots\right)\times$$
$$\left(\frac{(-\mu'\sqrt{-1})^n}{[n]}+\frac{(-\mu'\sqrt{-1})^{n-2}}{2.[n-2](2n-1)}+\ldots\right).$$

To determine K_n for any particular value of n, we refer to the expression from which the two series were deduced; namely,

$$\left\{r^2+r'^2-2rr'\left(\mu\mu'+\sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\varphi-\varphi')\right)\right\}^{-\frac{1}{2}}.$$

When μ and μ' are each 1, then $P_n=1$, which gives an equation to find K_n.

8. Returning to the theory of attractions, we have, when the particle is internal,

$$V = \int_r^R \int_{-1}^{1} \int_0^{2\pi} \varrho\, r' \left(P_0 + P_1 \frac{r}{r'} + \ldots + P_n \frac{r^n}{r'^n} + \ldots\right) dr'\, d\mu'\, d\phi'$$

$$= 2\pi \int_r^R \int_{-1}^{1} \varrho \left(P_0 r' + P_1 r + P_2 \frac{r^2}{r'} + \ldots + P_n \frac{r^n}{r'^{n-1}} + \ldots\right) dr'\, d\mu',$$

for that portion of the body which is comprised between a sphere of radius r, and the surface of the body, supposed to be a surface of revolution round the axis of z; for, in that case R, the value of r' at the surface, is independent of ϕ'.

9. Suppose, for example, we wish to find the attraction of a homogeneous spheroid, on a point within it. In this case ϱ is constant, and

$$R^2 = \left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{-1};$$

a being the semi-major, and c the semi-minor axis.

First, all the even terms vanish; for the general even term is

$$2\pi\varrho \int_r^R \int_{-1}^{1} P_{2n+1} \frac{r^{2n+1}}{r'^{2n}}\, dr'\, d\mu' = -\frac{2\pi\varrho r^{2n+1}}{2n-1} \int_{-1}^{1} P_{2n+1} \left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{\frac{2n-1}{2}} d\mu'$$

$$+ \frac{2\pi\varrho r^2}{2n-1} \int_{-1}^{1} P_{2n+1}\, d\mu'.$$

Now P_{2n+1} consists of odd powers of μ'; and $\left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{\frac{2n-1}{2}}$ can be expanded in even powers of μ'; therefore the integral of the product (which is an odd function), taken from $\mu' = -1$ to $\mu' = 1$, is 0. Also $\int_{-1}^{1} P_{2n+1}\, d\mu' = 0$ *. All the odd terms above the third vanish; for the $(2n+1)$th term is

$$2\pi\varrho \int_r^R \int_{-1}^{1} P_{2n} \frac{r^{2n}}{r'^{2n-1}}\, dr'\, d\mu' = -\frac{\pi\varrho}{n-1} \int_{-1}^{1} P_{2n} r^{2n} \left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{n-1} d\mu'$$

$$+ \frac{\pi\rho}{n-1} \int_{-1}^{1} P_{2n} r^2\, d\mu', \text{ for it may be shown that } \int_{-1}^{1} \int_0^{2\pi} P_i P_{i'}\, d\mu'\, d\phi' = 0, \text{ if } i \text{ and } i'$$

be different integers. Now when n is greater than 1, $\left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{n-1}$ is a rational and entire function of μ', and, therefore, capable of being expressed in a series of LAPLACE'S coefficients †, the highest of which will be of the $(2n-2)$th order; and therefore no term of this expansion can be of the same order as P_{2n}; and the integral of the product of any two of different orders, between these limits, vanishes. So the second member of this vanishes.

* See Pratt. Mec. Phil., § 180.

† See Pratt. Mec. Phil., § 176. POISSON, Théorie Math. de la Chal., chap. viii. LAPLACE, Méc. Cél. liv. iii. chap. ii.

The first term is

$$2\pi\rho\int_r^R\int_{-1}^{1} r'\,dr'\,d\mu' = \pi\rho\int_{-1}^{1}\left(\frac{1-\mu'^2}{a^2}+\frac{\mu'^2}{c^2}\right)^{-1} d\mu' - \pi\rho r^2\int_{-1}^{1} d\mu'$$

$$= 2\pi\rho\left(a^2\frac{\sqrt{1-e^2}}{e} - r^2\right), \text{ (e being the eccentricity).}$$

The third term is

$$\frac{\pi}{2}K_2\rho\int_r^R\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)\left(\frac{1}{3}-\mu'^2\right)\frac{r^2}{r'}\,dr'\,d\mu'$$

$$= -\frac{\pi\rho}{4}K_2\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)\left(\frac{1}{3}-\mu'^2\right)r^2\log\left(\frac{1-\mu'^2}{a^2}+\frac{\mu'^2}{c^2}\right)d\mu',$$

$$= -\pi\rho\frac{K_2}{2}\left(\frac{1}{3}-\mu^2\right)r^2\left(\frac{2}{3}\frac{ca^2}{(a^2-c^2)^{\frac{3}{2}}}\tan^{-1}\frac{\sqrt{(a^2-c^2)}}{c} - \frac{2}{3}\frac{c^2}{a^2-c^2} - \frac{4}{9}\right),$$

$$= -3\pi\rho\left(\frac{1}{3}-\mu^2\right)r^2\left(\frac{1}{3}-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right), \text{ for } K_2 = 9.$$

The value of V for the sphere of radius r, calculated by the usual method, is $\frac{4\pi\rho r^2}{3}$*; consequently, for the whole ellipsoid, the value of V is

$$\frac{4\pi\rho r^2}{3} + 2\pi\rho\left(a^2\frac{\sqrt{(1-e^2)}}{e}\sin^{-1}e - r^2\right) - 3\pi\rho\left(\frac{1}{3}-\mu^2\right)r^2\left(\frac{1}{3}-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right).$$

Differentiate to f, by means of the equations $r^2 = f^2 + g^2 + h^2$ and $\mu = \frac{h}{r}$, and we have

$$-\frac{dV}{df} = \text{attraction in } x = +2\pi\rho f\left(1-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right);$$

$$\text{so } -\frac{dV}{dg} = \text{attraction in } y = +2\pi\rho g\left(1-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right);$$

$$\text{so } -\frac{dV}{dh} = \text{attraction in } y = +4\pi\rho h\left(+\frac{1}{e^2}-\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right);$$

which are the common expressions otherwise found†.

Also

$$-\frac{dV}{dr} = \text{attraction to centre} = +4\pi\rho\left\{\frac{r}{3}+\frac{3}{2}r\left(\frac{1}{3}-\mu^2\right)\left(\frac{1}{3}-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right)\right\}.$$

10. By a similar process, I have deduced the attraction of an oblate spheroid, on a point within it; the density varying inversely as the distance from the centre. The corresponding expressions are

$$-\frac{dV}{df} = +2\pi\rho\frac{f}{r} - \frac{3}{4}\frac{\pi\rho f}{a(1-e^2)}\left\{\frac{1}{e^2}-\frac{1}{3}-\left(\frac{(1-e^2)^2}{2e^3}+\frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\};$$

$$-\frac{dV}{dg} = +2\pi\rho\frac{g}{r} - \frac{3}{4}\frac{\pi\rho g}{a(1-e^2)}\left\{\frac{1}{e^2}-\frac{1}{3}-\left(\frac{(1-e^2)^2}{2e^3}+\frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\};$$

* See Pratt. Mec. Phil., § 172.

† Ibid. § 158.

$$-\frac{dV}{dh}=+2\pi\rho\frac{h}{r}+\frac{3}{2}\frac{\pi\rho h}{a(1-e^2)}\left\{\frac{1}{e^2}-\frac{1}{3}-\left(\frac{(1-e^2)^2}{2e^3}+\frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\};$$

$$-\frac{dV}{dr}=+2\pi\rho+\frac{3}{4}\frac{\pi\rho(3\mu^2-1)}{a(1-e^2)}\left\{\frac{1}{e^2}-\frac{1}{3}-\left(\frac{(1-e^2)^2}{2e^3}+\frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\}.$$

11. When the particle attracted is external, the series in (3.) does not give a finite expression. Instead of taking it separately, I make it a case of a general theorem which follows.

12. To find the attraction of a spheroid on a point within it, the density being any function of the distance from the centre, and the eccentricity being small.

Let $\rho\,\varphi\,r'$ represent the law of density; then the value of V, for the portion comprised between the surface and a sphere of radius r, is

$$2\pi\rho\int_r^R\int_{-1}^1\varphi r'.r'\left(P_0+P_1\frac{r}{r'}+\ldots+P_n\frac{r^n}{r'^n}+..\right)dr'\,d\mu'.$$

Integrating by parts we have

$$\int\varphi r'.r'^{1-n}.dr'=\frac{\Phi_I r'}{r'^{n-1}}+(n-1)\frac{\Phi_{II}r'}{r'^n}+(n-1)\,n\frac{\Phi_{III}r'}{r'^{n+1}}+(n-1)n(n+1)\frac{\Phi_{IV}r'}{r'^{n+1}}+\&c.$$

Therefore the $(n+1)$th term of V is

$$2\pi\rho r^n\int_{-1}^1 P_n\left(\frac{\Phi_I R}{R^{n-1}}+(n-1)\frac{\Phi_{II}R}{R^n}+(n-1)\,n\frac{\Phi_{III}R}{R^{n+1}}+(n-1)n(n+1)\frac{\Phi_{IV}R}{R^{n+2}}+..\right)d\mu'$$

$$-2\pi\rho r^n\int_{-1}^1 P_n\left(\frac{\Phi_I r}{r^{n-1}}+(n-1)\frac{\Phi_{II}r}{r^n}+(n-1)\,n\frac{\Phi_{III}r}{r^{n+1}}+(n-1)n(n+1)\frac{\Phi_{IV}r}{r^{n+2}}+\ldots\right)d\mu'.$$

Now $R=\left\{\frac{1}{a^2}+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\mu'^2\right\}^{-\frac{1}{2}}=a\left(1+e^2\mu'^2\right)^{-\frac{1}{2}}=a\left(1-\frac{e^2}{2}\mu'^2\right)$, rejecting e^4 and higher powers of e; and $\varphi_{(m)}R=\varphi_{(m)}a-\varphi_{(m-1)}a.\frac{ae^2\mu'^2}{2}$ to the same degree of accuracy. The last member of the expression for V need only be calculated when $n=0$; for all the rest of the terms $\left(\text{involving}\int_{-1}^1 P_n\,d\mu'\text{ where }n>0\right)$ vanish.

The first member need only be calculated when $n=0$, and when $n=2$; for when n is odd, it vanishes as before; and also when n is even and greater than 2: for the functions of R involve no higher powers of μ' than the square; and consequently they vanish, when multiplied by P_4, P_6,.. &c., and integrated with respect to μ', from -1 to $+1$*.

When $n=0$, the term is

$$2\pi\rho\int_{-1}^1(R\varphi_I R-\varphi_{II}R)\,d\mu'-2\pi\rho\int_{-1}^1(r\varphi_I r-\varphi_{II}r)\,d\mu',$$

$$=2\pi\rho\int_{-1}^1\left(a\varphi_I a-\varphi a.\frac{a^2e^2\mu'^2}{2}-\varphi_{II}a\right)d\mu'-4\pi\rho(r\varphi_I r-\varphi_{II}r),$$

$$=4\pi\rho\left(a\varphi_I a-\varphi_{II}a-\frac{e^2}{6}a^2\varphi a-(r\phi_I r-\phi_{II}r)\right).$$

* See Art. 9.

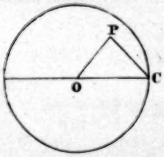
When $n = 2$, the term is

$$\frac{9}{2}\pi\rho r^2\left(\mu^2-\frac{1}{3}\right)\int_{-1}^{1}\left(\mu'^2-\frac{1}{3}\right)\left(\frac{\phi_1 R}{R}+\frac{\phi_{11} R}{R^2}+2\,\frac{\phi_{111} R}{R^3}+2.3\,\frac{\phi_{1v} R}{R^4}+\ldots\right)d\mu'$$

$$=\frac{9}{2}\pi\rho r^2\left(\mu^2-\frac{1}{3}\right)\Sigma_1^{\infty}[m-1]\int_{-1}^{1}\left(\mu'^2-\frac{1}{3}\right)\frac{\phi_m R}{R^m}\,d\mu',$$

$$=\frac{9}{2}\pi\rho r^2\left(\mu^2-\frac{1}{3}\right)\Sigma_1^{\infty}[m-1]\int_{-1}^{1}\left(\mu'^2-\frac{1}{3}\right)\left(\frac{\phi_m a}{a^m}-\frac{\phi_{m-1}a}{a^{m-1}}\cdot\frac{e^2\mu'^2}{2}+\frac{\phi_m a}{a^m}\cdot\frac{m\,e^2\mu'^2}{2}\right)d\mu',$$

$$=\frac{2}{5}\pi\rho r^2\left(\mu^2-\frac{1}{3}\right)e^2\,\Sigma_1^{\infty}[m-1]\left(\frac{\phi_m a}{a^m}\cdot m-\frac{\phi_{m-1}a}{a^{m-1}}\right)=-\frac{2}{5}\pi\rho r^2\left(\mu^2-\frac{1}{3}\right)e^2\,\varphi a.$$



To calculate V for the sphere whose radius is r.

Let the sphere be referred to polar coordinates, the centre being the pole.

Let $OC = r$, $OP = r_1$, and $POC = \theta$; then $PC = \sqrt{(r^2 + r_1^2 - 2rr_1\cos\theta)}$.

Mass of the element at $P = \rho r_1^2\,\phi r_1\,dr_1\sin\theta\,d\theta\,d\omega$, and

$$V=\int_0^r\int_0^{\pi}\int_0^{2\pi}\frac{\rho r_1^2\phi r_1\,dr_1\sin\theta\,d\theta\,d\omega}{\sqrt{(r^2+r_1^2-2rr_1\cos\theta)}}=2\pi\rho\int_0^r\int_0^{\pi}\frac{r_1^2\phi r_1\sin\theta\,dr_1\,d\theta}{\sqrt{(r^2+r_1^2-2rr_1\cos\theta)}}$$

$$=2\pi\rho\int_0^r\frac{r_1\phi_1 r_1}{r}\Big((r+r_1)-(r-r_1)\Big)\,dr_1=\frac{4\pi\rho}{r}\int_0^r r_1^2\,\phi r_1\,dr_1$$

$$=\frac{4\pi\rho}{r}\left(r^2\,\phi_1 r-2r\,\phi_{11}r+2\,\phi_{111}r-\mathrm{K}\right),$$

K being the value of $r_1^2\,\phi_1 r_1 - 2r_1\,\phi_{11}r_1 + 2\,\phi_{111}r_1$, when $r_1 = 0$.

The whole value of V then is

$$4\pi\rho\left\{a\,\phi_1 a-\phi_{11}a-\frac{1}{6}e^2a^2\phi a-\left(\phi_{11}r-\frac{2}{r}\phi_{111}r+\frac{\mathrm{K}}{r}\right)-\frac{1}{10}r^2\left(\mu^2-\frac{1}{3}\right)e^2\,\phi a\right\}.$$

And the attraction toward the centre

$$=-\frac{dV}{dr}=4\pi\rho\left\{\frac{\psi r}{r^2}+\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)r\,e^2\,\phi a\right\},$$

where

$$\psi r=2\,\phi_{111}r-2r\,\phi_{11}r+r^2\,\phi_1 r-\mathrm{K}=\int_0^r r^2\,\phi r\,dr.$$

13. To find the attraction of the same spheroid on a particle without it.

The series (Art. 3.) is

$$2\pi\rho\int_0^R\int_{-1}^{1}\phi r'\left(\mathrm{P}_0\frac{r'^0}{r}+\mathrm{P}_1\frac{r'^1}{r^2}\cdot+\ldots\right)dr'\,d\mu'.$$

Now

$$\int\phi r'\,.\,r'^{n+1}\,.\,dr'=r'^{n+1}\,\phi_1 r'-(n+1)\,r'^n\,\phi_{11}r'+n(n+1)\,r'^{n-1}\,\phi_{111}r'-\&c.\ (\mathrm{A});$$

and the general (nth) term of V is

$$\frac{2\pi\rho}{r^n}\int_{-1}^{1}\mathrm{P}_{n-1}\left(r'^{n+1}\,\varphi_1 r'-(n+1)\,r'^n\,\phi_{11}r'+n(n+1)\,r'^{n-1}\,\phi_{111}r'-\&c.\right)d\mu'$$

$$-\frac{2\pi\rho}{r^n}\int_{-1}^{1}\mathrm{P}_{n-1}\,\mathrm{C}\,d\mu',$$

where C is the value of (A) when $r' = 0$. As before, the last term need be calculated only when $n = 1$; and the first when $n = 1$, and $n = 3$.

When $n = 1$, it is

$$\frac{2\pi\rho}{r}\int_{-1}^{1}(R^2\phi_i R - 2R\phi_{ii}R + 2\phi_{iii}r - K)\,d\mu' = \frac{2\pi\rho}{r}\int_{-1}^{1}\psi r\,d\mu'$$

$$= \frac{4\pi\rho}{r}\left(\psi a - \frac{e^2}{6}a^3\phi a\right).$$

When $n = 3$, it is

$$\frac{9}{2}\left(\mu^2 - \frac{1}{3}\right)\frac{\pi\rho}{r^3}\int_{-1}^{1}d\mu'\left(\mu'^2 - \frac{1}{3}\right)\left(R^4\phi_i R - 4R^3\phi_{ii}R + 12R^2\phi_{iii}R - 24R\phi_{iv}R\right.$$

$$\left. + 24\phi_v R\right) = -\frac{2}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{\pi\rho a^5}{r^3}\phi a\,.\,e^2.$$

The whole value of V is

$$\frac{4\pi\rho}{r}\left\{\psi a - \frac{e^2}{6}a^3\phi a - \frac{1}{10}\left(\mu^2 - \frac{1}{3}\right)\frac{a^5}{r^3}\phi a\,.\,e^2\right\}.$$

And

$$-\frac{dV}{dr} = \frac{4\pi\rho}{r^2}\left\{\psi a - \frac{e^2}{6}a^3\phi a - \frac{3}{10}\left(\mu^2 - \frac{1}{3}\right)\frac{a^5}{r^3}\phi a\,.\,e^2\right\}.$$

14. Instead of the eccentricity $e = \sqrt{\left(1 - \frac{c^2}{a^2}\right)}$ it will be more convenient to employ the ellipticity $\varepsilon = 1 - \frac{c}{a}$*. These give $\varepsilon = \frac{e^2}{2}$. And the values of V become for an internal point,

$$4\pi\rho\left\{a\phi_i a - \phi_{ii}a - \left(\phi_{ii}r - \frac{2}{r}\phi_{iii}r + \frac{K}{r}\right) - \frac{\varepsilon}{3}a^2\phi a - \frac{r^2}{5}\left(\mu^2 - \frac{1}{3}\right)\varepsilon\phi a\right\};$$

for an external point,

$$4\pi\rho\frac{1}{r}\left\{\psi a - \frac{\varepsilon}{3}a^3\phi a - \frac{3}{5}\frac{1}{r^3}\left(\mu^2 - \frac{1}{3}\right)a^5\phi a\right\}.$$

15. To find the attraction on the supposition that the body is composed of spheroidal layers, homogeneous in themselves, but differing from one another in density and ellipticity.

First, on an internal point.

Let r', as before, be the radius vector of any layer; a' its equatorial radius; $\rho\,\phi\,a'$ its density, and ε' its ellipticity, being some function of a' as $\chi\,a'$. Then

$$a' = r'(1 + \varepsilon'\mu'^2) \text{ and } \phi\,a' = \phi\,r' + r'\phi'r'\chi r'\mu'^2 = \phi\,r' + Fr'.\mu'^2, \text{ suppose.}$$

Consequently to the term before produced in (12.) by $\varphi\,r'$ we must add a term similarly produced by $F\,r'.\mu'^2$. Also, instead of taking, in the first instance, the portion comprised between the surface and a sphere of radius r, we must take the por-

* See Puissant, vol. i. p. 259, where the word ellipticity is used in this sense.

tion between the surface and that spheroid on which the point lies, whose ellipticity is ϵ_1. The $(n+1)$th term of V now becomes

$$2\pi\rho r^n \int_{-1}^{1} P_n \, d\mu' \left\{ \int_r^R \frac{\phi r'}{r'^{n-1}} \, dr' + \mu'^2 \int_r^R \frac{F r'}{r'^{n-1}} \, dr' \right\}.$$

The first part of this gives $4\pi\rho$

$$\left\{ \begin{array}{l} a\,\phi_i a - \phi_{ii} a - \frac{\epsilon}{3} a^2 \phi a - \frac{\epsilon}{5}\left(\mu^2 - \frac{1}{3}\right) r^2 \phi a \\ -\left(a\,\phi_i a - \phi_{ii} a - \frac{\epsilon_1}{3} a^2 \phi a - \frac{\epsilon_1}{5}\left(\mu^2 - \frac{1}{3}\right) r^2 \phi a\right) \end{array} \right\},$$

a being semiaxis major of the stratum on which r lies. To determine the other part, it is necessary to compute it when $n = 0$ and $n = 2$, which gives

$$4\pi\rho \left\{ \frac{1}{3} \int_a^a a' F a' \, da' + \frac{1}{5} r^2 \left(\mu^2 - \frac{1}{3}\right) \int_a^a \frac{F a'}{a'} \, da' \right\}.$$

To the sum of these we must add the value of V for the inner spheroid; and for this purpose we have to obtain V for an external point.

To the expression in (13.) we must add

$$\frac{2\pi\rho}{r^n} \int_{-1}^{1} d\mu' \left(\mu'^2 P_{n-1} \int_0^R F r' r'^{n+1} \, dr'\right),$$

to be calculated when $n = 1$ and $n = 3$. This is

$$4\pi\rho \left\{ \frac{1}{3r} \int_0^a F a' \,.\, a'^2 \, da' + \frac{\mu^2 - \frac{1}{3}}{5r^3} \int_0^a F a' \,.\, a'^4 \, da' \right\}.$$

The whole value of V is

$$\frac{4\pi\rho}{r} \left\{ \begin{array}{l} \psi a - \frac{\epsilon}{3} a^3 \phi a - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right) \epsilon \frac{a^5}{r^3} \phi a + \frac{1}{3} \int_0^a F a' \,.\, a'^2 \, da' \\ \qquad + \frac{1}{5} \frac{\mu^2 - \frac{1}{3}}{r^3} \int_0^a F a' \,.\, a'^4 \, da' \end{array} \right\}.$$

After writing a for a, and ϵ_1 for ϵ, add this to the other value of V, and apply the equations

$$\int_a^a F a' \,.\, a' \, da' = \int_a^a a'^2 \epsilon' \, d\phi a' = a^2 \epsilon \phi a - a^2 \epsilon_1 \phi a - \int_a^a \phi a' \, d(a'^2 \epsilon'),$$

$$\frac{1}{r} \int_0^a F a' \,.\, a'^2 \, da' = \frac{1}{r} \int_0^a a'^3 \epsilon' \, d\phi a' = a^2 \epsilon_1 \phi a - \frac{1}{r} \int_0^a \phi a' \, d(a'^3 \epsilon'),$$

and similar equations for the other integrals; and we shall obtain

$$V = 4\pi\rho \left\{ \begin{array}{l} a\phi_i a - \phi_{ii} a - a\phi_i a + \phi_{ii} a + \frac{\psi a}{r} - \frac{1}{3}\left\{ \int_a^a \phi a' \, d(a'^2 \epsilon') + \frac{1}{r} \int_0^a \phi a' \, d(a'^3 \epsilon') \right\} \\ \qquad - \frac{r^2}{5}\left(\mu^2 - \frac{1}{3}\right)\left\{ \int_a^a \phi a' \, d\epsilon' + \frac{1}{r^3} \int_0^a \phi a' \, d(a'^5 \epsilon') \right\} \end{array} \right\}.$$

16. To find the equation of equilibrium of a heterogeneous spheroidal mass of fluid, revolving about its axis, with an angular velocity ω.

By the principles of hydrostatics the general equation is $\int \frac{dp}{\varrho'} = \int (X\,dx + Y\,dy + Z\,dz)$, ϱ' being the density at the point (x, y, z), p the pressure, and X, Y, Z the sums of the resolved parts of the forces; which are $-\frac{dV}{dx}$, $-\frac{dV}{dy}$, $-\frac{dV}{dz}$, and the centrifugal force. Let the axis of z be that of rotation; then the centrifugal force is $\omega^2 x$ along x, and $\omega^2 y$ along y. Let us express ω in terms of the ratio of the centrifugal force at the equator to the equatorial gravity. Call this ratio m, which is small in the case of the earth, being of the same order as ε. Then

$$m = \frac{\omega^2 a^3}{\text{mass}} = \frac{\omega^2 a^3}{4\pi\varrho\psi a}, \text{ or } \omega^2 = \frac{4\pi\varrho m\psi a}{a^3}.$$

Therefore

$$X = \frac{dV}{dx} + \frac{4\pi\varrho m\psi a . x}{a^3},\ Y = \frac{dV}{dy} + \frac{4\pi\varrho m\psi a . y}{a^3},\ Z = \frac{dV}{dz},$$

and

$$\int \frac{dp}{\varrho'} = V + \frac{2\pi\varrho m\psi a}{a^3}(1-\mu^2)\,r^2.$$

Now $\int \frac{dp}{\rho'}$ is a constant for a level surface. Hence for any stratum we have

$$C = V + \frac{2\pi\rho m\psi a}{a^3}(1-\mu^2)\,r^2.$$

At the surface this is

$$C = \frac{\psi a}{r} - \frac{1}{3r}\int_0^a \phi\,a'\,d\,(a'^3\varepsilon') - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{1}{r^3}\int_0^a \phi\,a'\,d\,(a'^5\varepsilon') + \frac{m}{2}\frac{\psi a}{a^3}(1-\mu^2)r^2,$$

$$= \frac{1}{3r}\,M\,a - \frac{1}{5r^3}\left(\mu^2 - \frac{1}{3}\right)N\,a + \frac{1}{3}\frac{m r^3}{2a^3}\,M\,a\,(1-\mu^2),$$

where

$$M\,a = \int_0^a \phi\,a'\,\frac{d\,(a'^3(1-\varepsilon'))}{d\,a'}\,d\,a', \text{ and } N\,a = \int_0^a \phi\,a'\,\frac{d\,(a'^5\varepsilon')}{d\,a'}\,d\,a'.$$

For r write $a\,(1-\varepsilon\mu^2)$, then

$$C = \frac{M\,a}{3\,a}(1+\varepsilon\mu^2) - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{N\,a}{a^3} + \frac{1}{6}\,m\,(1-\mu^2)\,\frac{M\,a}{a}.$$

Equate the coefficients of μ^2, then

$$\frac{M\,a}{a}\left(\varepsilon - \frac{m}{2}\right) = \frac{3\,N\,a}{5\,a^3} \quad \ldots\ (B).$$

17. By differentiating and changing the sign of $\int \frac{dp}{\rho'}$, we obtain the amount of gravity which acts towards the centre; which, to the order we are now considering, is the same as the whole force of gravity; since the cosine of the angle of the vertical differs from unity only by terms of a higher order.

Consequently

$$g = 4\pi\varrho\left\{\frac{M\,a}{3\,r^2} - \frac{3}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{N\,a}{r^4} - \frac{m}{3}(1-\mu^2)\frac{r\,M\,a}{a^3}\right\},$$

$$= 4\pi\varrho\left\{\frac{M\,a}{3\,a^2}(1 + 2\,\varepsilon\,\mu^2) - \frac{3}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{N\,a}{a^4} - \frac{m}{3}(1-\mu^2)\frac{M\,a}{a^2}\right\},$$

$$= 4\pi\varrho\,\frac{M\,a}{3\,a^2}\left\{1 + 2\,\varepsilon\,\mu^2 + (1 - 3\,\mu^2)\left(\varepsilon - \frac{m}{2}\right) - m\,(1-\mu^2)\right\}\text{(by B.)},$$

$$= 4\pi\varrho\,\frac{M\,a}{3\,a^2}\left\{1 + \varepsilon - \frac{3\,m}{2} + \mu^2\left(\frac{5}{2}\,m - \varepsilon\right)\right\} = G\left\{1 + \sin^2 l\left(\frac{5}{2}\,m - \varepsilon\right)\right\},$$

where G is the equatorial gravity and l the latitude.

18. From this it appears that up to terms of the 1st order, $R = a\,(1 - \varepsilon \sin^2 l)$ is the equation of the curve which generates the surface of equilibrium, where the value of ε depends on m, or on the velocity of rotation: but as the coefficients of higher powers of sin l may be considerable, it will be useful to find the surface of equilibrium to a greater degree of exactness. For this purpose we must introduce the fourth power of sin l, whose coefficient will be of the second order. Let the equation of the strata be $a' = r'\,(1 + \varepsilon'\,\mu^2 + A'\,\mu^4)$, or $r' = a'\left(1 - \varepsilon'\,\mu^2 + (\varepsilon'^2 - A')\,\mu^4\right)$, where ε' and A' are functions of a' as $\chi\,a'$ and $\theta\,a'$. Then

$$\phi\,a' = \phi\,r' + F\,r'\,\mu^2 + r'^2\frac{d\,(\chi\,r'^3\,.\,\phi'\,r)}{d\,r}\frac{\mu^4}{2} + r\,\phi'\,r\,\theta\,r\,\mu^4 = \phi\,r + F\,r\,.\,\mu^2 + \Pi\,r\,.\,\mu^4, \text{ suppose.}$$

The $(n+1)$th term of V for an internal point now becomes

$$2\pi\varrho\,r^n\int_{-1}^{1} d\mu'\,P_n\left\{\int_r^R \frac{\phi\,r'}{r'^{n-1}}\,dr' + \mu'^2\int_r^R \frac{F\,r'}{r'^{n-1}}\,dr' + \mu'^4\int_r^R \frac{\Pi\,r'}{r'^{n-1}}\,dr'\right\}.$$

By writing for r and R their values in terms of a and a, it is easily found that

$$\int_r^R \frac{\phi\,r'}{r'^{n-1}}\,dr' \text{ equals } \int_0^a \frac{\phi\,a'}{a'^{n-1}}\,da' + \left(\frac{\phi\,a}{a^{n-2}}\right)\left(-\varepsilon\,\mu'^2 + \mu'^4(\varepsilon^2 - A') - \mu'^4\frac{n-1}{2}\,\varepsilon^2\right)$$

$$+ \left(\frac{\phi'\,a}{a^{n-3}}\right)\varepsilon^2\frac{\mu'^4}{2} - \text{the same functions of a and } \varepsilon_1.$$

And that similar equations are true for the two remaining integrals.

1st. Let $n = 0$, and we have

$$2\pi\varrho\int_{-1}^{1} d\mu'\left\{\begin{array}{l}\int_0^a a'\,\phi\,a'\,da' + a^2\,\phi\,a\left(-\varepsilon\,\mu'^2 + \mu'^4\left(\frac{3}{2}\,\varepsilon^2 - A\right)\right) + a^3\,\phi'\,a\,\frac{\varepsilon^2\,\mu'^4}{2} \\ + \mu'^2\int_0^a a'\,.\,F\,a'\,da' + \mu'^2\,a^2\,F\,a\,(-\varepsilon\,\mu'^2) + \mu'^4\int_0^a a'\,\Pi\,a'\,da' \\ - \text{same function of a and } \varepsilon_1,\end{array}\right\},$$

which gives

$$4\pi\varrho\left\{\int_0^a a'\,\phi\,a'\,da' + a^2\,\phi\,a\left(-\frac{\varepsilon}{3} + \frac{1}{5}\left(\frac{3}{2}\,\varepsilon^2 - A\right)\right) + a^3\,\phi'\,a\,\frac{\varepsilon^2}{10} + \frac{1}{3}\int_0^a F\,a'\,.\,a'\,.\,da'\right.$$

$$\left. - a^2\,F\,a\,\frac{\varepsilon}{5} + \frac{1}{5}\int_0^a a'\,.\,\Pi\,a'\,da'\right\} - 4\pi\varrho\,(\text{same function of } a_1\,\varepsilon_1 \text{ and } A_1).$$

2nd. Let $n = 2$, and we get

$$\frac{9}{4} 2\pi\varrho\left(\mu^2 - \frac{1}{3}\right) r^2 \int_{-1}^{1} d\mu' \left(\mu'^2 - \frac{1}{3}\right)\left\{\int_a^a \frac{\phi a'}{a'} da' + (\phi a - \phi a)\left(-\varepsilon' \mu'^2 + \mu'^4\left(\frac{\varepsilon'^2}{2} - A'\right)\right)\right.$$

$$\left. + (a\phi' a - a\phi' a)\frac{\varepsilon'^2 \mu'^4}{2} + \int_a^a \frac{F a'}{a'} da' + (F a - F a)(-\varepsilon' \mu'^2) + \int_a^a \Pi a' . a' . da'\right\},$$

which gives

$$4\pi\varrho\left(\mu^2 - \frac{1}{3}\right) r^2 \left\{\phi a\left(-\frac{\varepsilon}{5} + \frac{6}{35}\left(\frac{\varepsilon^2}{2} - A\right)\right) + a\phi' a \frac{3}{35}\varepsilon^2 + \frac{1}{5}\int_0^a \frac{F a'}{a'} da' - \frac{6}{35} F a . \varepsilon\right.$$

$$\left. + \frac{6}{35}\int_0^a \frac{\Pi a'}{a'} da'\right\} - 4\pi\varrho\left(\mu^2 - \frac{1}{3}\right) r^2 \text{ (same function of a, } \varepsilon_1 \text{ and } A_1).$$

3rd. Let $n = 3$, and we have

$$\frac{1225}{64}\left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right) 2\pi\varrho r^4 \int_{-1}^{1} d\mu' \left(\mu'^4 - \frac{6}{7}\mu'^2 + \frac{3}{35}\right)\left\{\int_a^a \frac{\phi a'}{a'^3} da' + \left(\frac{\phi a}{a^3} - \frac{\phi a}{a^3}\right)\right.$$

$$\left(-\varepsilon' \mu'^2 - \mu'^4\left(\frac{\varepsilon'^2}{2} + A'\right)\right) + \left(\frac{\phi' a}{a} - \frac{\phi' a}{a}\right)\varepsilon'^2 \frac{\mu'^4}{2} + \int_a^a \frac{F a'}{a'^3} da' + \left(\frac{F a}{a^3} - \frac{F a}{a^3}\right)(-\varepsilon' \mu'^2)$$

$$\left. + \int_a^a \frac{\Pi a'}{a'^3} da'\right\},$$

or

$$\left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right) 4\pi\varrho r^4 \left\{-\frac{\phi a}{a^3}\frac{1}{9}\left(\frac{\varepsilon^2}{2} + A\right) + \frac{\phi' a}{a}\frac{\varepsilon^2}{18} - \frac{\varepsilon}{9}\frac{F a}{a^3} + \frac{1}{9}\int_0^a \frac{\Pi a'}{a'^3} da'\right\}$$

$$- \left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right) 4\pi\varrho r^4 \text{ (same function of a, } \varepsilon_1 \text{ and } A_1).$$

19. To the sum of these must be added V for the inner spheroid, for which we shall have to find the value of V generally, when the attracted point is without the body.

The general term of the series for V is

$$\frac{2\pi\varrho}{r^{n-1}} \int_{-1}^{1} d\mu' P_{n-2}\left\{\int_0^R r'^n . \phi r' . dr' + \mu'^2 \int_0^R r'^n F r' . dr' + \mu'^4 \int_0^R r'^n . \Pi r' . dr'\right\},$$

and

$$\int_0^R r'^n \phi r' dr' = \int_0^a a'^n . \phi a' . da' + a^{n+1}\phi a\left(-\varepsilon\mu^2 + \mu^4(\varepsilon^2 - A) + \mu^4 \frac{n}{2}\varepsilon^2\right)$$

$$+ a^{n+2}\phi' a \frac{\varepsilon^2 \mu^4}{2};$$

and similar equations hold for the other functions.

1st. Let $n = 2$, then we have

$$\frac{2\pi\varrho}{r}\int_{-1}^{1} d\mu' \left\{\int_0^a a'^2 \phi a' da' + a^3 \phi a\left(-\varepsilon\mu'^2 + \mu'^4(2\varepsilon^2 - A)\right) + a^4\phi' a \frac{\varepsilon^2}{10}\right.$$

$$\left. + \int_0^a a'^2 F a' da' + a^3 F a(-\varepsilon\mu'^2) + \int_0^a a'^2 \Pi a' da'\right\},$$

which is equal to

$$\frac{4\pi\varrho}{r}\left\{\int_0^a a'^2 \phi a' da' + a^3 \phi a\left(-\frac{\varepsilon}{3} + \frac{1}{5}(2\varepsilon^2 - A)\right) + a^4\phi' a \frac{\varepsilon^2}{10} + \frac{1}{3}\int_0^a a'^2 F a' . da'\right.$$

$$\left. - \frac{1}{5} a^3 F a . \varepsilon + \frac{1}{5}\int_0^a a'^2 \Pi a' da'\right\}.$$

2nd. Let $n = 4$, and the term is

$$\frac{9}{4}\left(\mu^2-\frac{1}{3}\right)\frac{2\pi\rho}{r^3}\int_{-1}^{1} d\mu'\left(\mu'^2-\frac{1}{3}\right)\left\{\int_0^a a'^4\,\varphi a'\,da'\,da' + a^5\,\varphi a\left(-\varepsilon\mu'^2+\mu'^4(3\varepsilon^2-\mathrm{A})\right)\right.$$
$$\left. + a^6\,\varphi' a\,\frac{\varepsilon^2\mu'^4}{2} + \int_0^a a'^4\,\mathrm{F}\,a'\,da'\,da' - a^5\,\mathrm{F}\,a\,\varepsilon\mu'^2 + \int_0^a a'^4\,\Pi\,a'\,da'\,da'\right\},$$

or

$$4\pi\rho\,\frac{\mu^2-\frac{1}{3}}{r^3}\left\{a^5\,\varphi\,a\left(-\frac{\varepsilon}{5}+\frac{6}{35}(3\varepsilon^2-\mathrm{A})\right) + a^6\,\varphi' a\,\frac{3}{35}\varepsilon^2 + \frac{1}{5}\int_0^a a'^4\,\mathrm{F}\,a'\,da'\right.$$
$$\left. - \frac{6}{35}a^5\,\mathrm{F}\,a\,.\,\varepsilon + \frac{6}{35}\int_0^a a'^4\,\Pi\,a'\,da'\,da'\right\}.$$

3rd. Let $n = 6$, and we get

$$\frac{1225}{64}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{4\pi\rho}{r^5}\int_{-1}^{1} d\mu'\left(\mu'^4-\frac{6}{7}\mu'^2+\frac{3}{35}\right)\left\{\int_0^a a'^6\,\varphi\,a'\,da'\,da'\right.$$
$$+ a^7\,\varphi\,a\left(-\varepsilon\mu'^2+\mu'^4(4\varepsilon^2-\mathrm{A})\right) + a^8\,\varphi' a\,\frac{\varepsilon^2\mu'^4}{2} + \int_0^a a'^6\,\mathrm{F}\,a'\,da'\,da' - a^7\,\mathrm{F}\,a\,\varepsilon\mu'^2$$
$$\left. + \int_0^a a'^6\,\Pi\,a'\,da'\,da'\right\},$$

which is

$$\frac{4\pi\rho}{r^5}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\left\{a^7\,\varphi\,a\,\frac{1}{9}(4\varepsilon^2-\mathrm{A}) + a^8\,\varphi' a\,\frac{1}{18}\varepsilon^2 - \frac{\varepsilon}{9}a^7\,\mathrm{F}\,a + \frac{1}{9}\int_0^a a'^6\,\Pi\,a'\,da'\,da'\right\}.$$

20. In these three terms write a for a, ε_1 for ε, and A_1 for A, and add them to the other value of V, and apply the equations

$$\int_0^a a'^2\,\Pi\,a'\,da' = \frac{1}{2}\int_0^a a'^4\,d(\chi\,a'^2\,\varphi'\,a') + \int_0^a a'^3\,\theta\,a'\,d(\varphi\,a') = \frac{1}{2}a^4\varepsilon^2\,\varphi'\,a + a^3\,\mathrm{A}\,\varphi\,a$$
$$- 2\int_0^a a'^3\,\chi\,a'^2\,.\,d\varphi\,a' - \int_0^a \varphi\,a'\,d(a'^3\,\mathrm{A}') = \frac{1}{2}a^4\varepsilon^2\,\varphi'\,a + a^3\,\mathrm{A}\,\varphi\,a - 2a^3\varepsilon^2\,\varphi\,a$$
$$+ 2\int_0^a \varphi\,a'\,d(a'^3\varepsilon'^2) - \int_0^a \varphi\,a'\,d(a'^3\,\mathrm{A}'),$$

and similar equations for the other integrals, and the value of V for an internal point becomes

$$4\pi\rho\left\{\begin{array}{l}\displaystyle\int_a^a \varphi a'\,\frac{d\left(a'^3\left\{\frac{1}{2}-\frac{1}{3}\varepsilon'+\frac{1}{5}\left(\frac{3}{2}\varepsilon'^2-\mathrm{A}'\right)\right\}\right)}{da'}\,da' + \frac{1}{3r}\int_0^a \varphi\,a'\,\frac{d\left(a'^3\left\{1-\varepsilon'+\frac{3}{5}(2\varepsilon'^2-\mathrm{A}')\right\}\right)}{da'}\,da' \\ \displaystyle -\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)\left(r^2\int_a^a \varphi\,a'\,\frac{d\left(\varepsilon'-\frac{3}{7}(\varepsilon'^2-2\mathrm{A}')\right)}{da'}\,da' + \frac{1}{r^3}\int_0^a \varphi a'\,\frac{d\left(a'^5\left\{\varepsilon'-\frac{6}{7}(3\varepsilon'^2-\mathrm{A}')\right\}\right)}{da'}\,da'\right) \\ \displaystyle +\frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\left(r^4\int_a^a \varphi a'\,\frac{d\frac{1}{a'^3}\left(-\frac{\varepsilon'^2}{2}-\mathrm{A}'\right)}{da'}\,da' + \frac{1}{r^5}\int_0^a \varphi a'\,\frac{d\left(a'^7(4\varepsilon'^2-\mathrm{A}')\right)}{da'}\,da'\right)\end{array}\right.$$
$$= 4\pi\rho\left\{\mathrm{M}'a - \mathrm{M}'\mathrm{a} + \frac{1}{3r}\mathrm{M}\,\mathrm{a} + \frac{1}{5}\left(\mu^2-\frac{1}{3}\right)\left(r^2\mathrm{N}'a - r^2\mathrm{N}'\mathrm{a} + \frac{1}{r^3}\mathrm{N}\,\mathrm{a}\right)\right.$$
$$\left. + \frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\left(r^4\mathrm{P}'a - r^4\mathrm{P}'\mathrm{a} + \frac{1}{r^5}\mathrm{P}\,\mathrm{a}\right)\right\},$$

where the functions M, N, P, &c. are substituted for the corresponding integrals.

21. The equation of equilibrium will be formed exactly as in (16.), except that the expression for the velocity must be found more accurately. By (17.), the force of gravity at the equator is $4\pi\rho\frac{Ma}{3a^3}\left(1+\varepsilon-\frac{3m}{2}\right)$; and the centrifugal force is $\omega^2 a$; therefore $m=\frac{\omega^2 a}{4\pi\rho\frac{Ma}{3a^3}\left(1+\varepsilon-\frac{3m}{2}\right)}$, and $\omega^2=4\pi\rho\frac{Ma}{3a^3}\left(m+m\varepsilon-\frac{3m^2}{2}\right)$.

The equation of equilibrium becomes

$$\int\frac{dp}{\rho}=C=V+2\pi\rho r^2(1-\mu^2)\frac{Ma}{3a^3}m\left(1+\varepsilon-\frac{3m}{2}\right),$$

which at the surface becomes

$$C=\frac{Ma}{3r}-\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)\frac{Na}{r^3}+\frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{Pa}{r^5}+\frac{1}{2}(1-\mu^2)r^2\frac{Ma}{3a^3}\left(m+m\varepsilon-\frac{3m^2}{2}\right),$$

$$=\frac{Ma}{3a}(1+\varepsilon\mu^2+A\mu^4)-\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)(1+3\varepsilon\mu^2)\frac{Na}{a^3}+\frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{Pa}{a^5}$$

$$+\frac{1-\mu^2}{2}(1-2\varepsilon\mu^2)\frac{Ma}{3a^3}\left(m+m\varepsilon-\frac{3m^2}{2}\right).$$

Equate the coefficients of μ^2 and those of μ^4, and we have two equations for determining ε and A, thus showing that equilibrium is possible. These are,

$$\frac{Ma}{3a}\left(\varepsilon-\frac{m}{2}-\frac{3}{2}m\varepsilon+\frac{3}{4}\mu^2\right)-\frac{Na}{a^3}\left(\frac{1}{5}-\frac{\varepsilon}{5}\right)-\frac{2}{21}\frac{Pa}{a^5}=0,$$

and

$$\frac{Ma}{3a}(A+m\varepsilon)-\frac{Na}{a^3}\frac{3\varepsilon}{5}+\frac{1}{9}\frac{Pa}{a^5}=0;$$

whence

$$\frac{Ma}{3a}\left(\varepsilon-\frac{m}{2}+\frac{6A}{7}+\frac{3}{4}m^2+\frac{m\varepsilon}{7}-\frac{11}{7}\varepsilon^2\right)=\frac{Na}{5a^3},$$

and

$$\frac{Ma}{3a}\left(A-3\varepsilon^2+\frac{5m\varepsilon}{2}\right)=-\frac{1}{9}\frac{Pa}{a^5}.$$

22. The resultant attraction in the direction of r, is obtained by differentiating $\int\frac{dp}{\rho}$, as found in (21.), and changing its sign. This produces

$$4\pi\rho\left\{\frac{Ma}{3r^3}-\frac{3}{5}\left(\mu^2-\frac{1}{3}\right)\frac{Na}{r^4}+\frac{5}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{Pa}{r^6}-r(1-\mu^2)\left(m+m\varepsilon-\frac{3m^2}{2}\right)\frac{Ma}{3a^3}\right\},$$

since the terms arising from differentiating the expression for V in (20.) with respect to a, cancel each other.

For r write its value, and arrange the result according to powers of μ, and this formula becomes

$$4\pi\rho\left\{\begin{array}{l}\frac{Ma}{3a^3}\left(1-m+m\varepsilon-\frac{3m^2}{2}+\mu^2\left(2\varepsilon+m+2m\varepsilon-\frac{3m^2}{2}\right)+\mu^4(2A+\varepsilon^2-m\varepsilon)\right)\\ \quad-\frac{Na}{a^4}\left(-\frac{1}{5}+\mu^2\left(\frac{3}{5}-\frac{4}{5}\varepsilon\right)+\mu^4\frac{12}{5}\varepsilon\right)+\frac{Pa}{a^6}\left(\frac{1}{21}-\mu^2\frac{10}{21}+\mu^4\frac{5}{9}\right)\end{array}\right\}$$

$$=4\pi\rho\frac{Ma}{3a^3}\left\{1+\varepsilon-\frac{3m}{2}-\frac{27}{14}m\varepsilon+\frac{9}{4}m^2-\frac{2}{7}\varepsilon^2+\frac{3}{7}A\right.$$

$$\left.+\mu^2\left(\frac{5}{2}m-\varepsilon+\frac{72}{7}m\varepsilon-\frac{15}{4}m^2-\frac{29}{7}\varepsilon^2+\frac{12}{7}A\right)+\mu^4\left(4\varepsilon^2-3A-\frac{15}{2}m\varepsilon\right)\right\},$$

by applying the two equations deduced in last article. The total attraction normal to the surface, or the force of gravity, is found from this by dividing by the cosine of the angle of the vertical, or by $1 - 2\varepsilon^2(\mu^2 - \mu^4)$; whence

$$g = 4\pi\varrho\frac{Ma}{3a^3}\left\{1 + \varepsilon - \frac{3m}{2} - \frac{27}{14}m\varepsilon + \frac{9}{4}m^2 - \frac{2}{7}\varepsilon^2 + \frac{3}{7}A\right.$$
$$\left. + \mu^2\left(\frac{5}{2}m - \varepsilon + \frac{72}{7}m\varepsilon - \frac{15}{4}m^2 - \frac{15}{7}\varepsilon^2 + \frac{12}{7}A\right) + \mu^4\left(2\varepsilon^2 - 3A - \frac{15}{2}m\varepsilon\right)\right\}.$$

Let λ be the sine of the real latitude; then as $\mu^2 = \lambda^2 - 4\varepsilon\lambda^2\overline{-\lambda^4}$, we get

$$g = 4\pi\varrho\frac{Ma}{3a^3}\left\{1 + \varepsilon - \frac{3m}{2} - \frac{27}{14}m\varepsilon + \frac{9}{4}m^2 - \frac{2}{7}\varepsilon^2 + \frac{3}{7}A\right.$$
$$\left. + \lambda^2\left(\frac{5}{2}m - \varepsilon + \frac{2}{7}m\varepsilon - \frac{15}{4}m^2 + \frac{13}{7}\varepsilon^2 + \frac{12}{7}A\right) - \lambda^4\left(2\varepsilon^2 + 3A - \frac{5}{2}m\varepsilon\right)\right\}.$$

Let G represent the equatorial gravity; then

$$g = G\left\{1 + \lambda^2\left(\frac{5}{2}m - \varepsilon - \frac{26}{7}m\varepsilon + \frac{20}{7}\varepsilon^2 + \frac{12}{7}A\right) - \lambda^4\left(2\varepsilon^2 + 3A - \frac{5}{2}m\varepsilon\right)\right\}$$
$$= G\left\{1 + \sin^2 l\left(\frac{5}{2}m - \varepsilon - \frac{17}{14}m\varepsilon + \frac{6}{7}\varepsilon^2 - \frac{9}{7}A\right) + \sin^2 l\cos^2 l\left(2\varepsilon^2 + 3A - \frac{5}{2}m\varepsilon\right)\right\},$$

which is an extension of CLAIRAUT's theorem.

23. In this process A indicates the amount of deviation of the required surface from the surface represented by $r = a(1 - \varepsilon\mu^2)$. If the equation had been taken $a = r\left\{1 + e\mu^2 - \left(\frac{3}{2}e^2 + B\right)\left(\mu^4 - \mu^2\right) + e^2\mu^4\right\}$, B would have been the index of deviation from an elliptic spheroid.

To apply CLAIRAUT's theorem to this surface, we have

$$\varepsilon = e + \frac{3}{2}e^2 + B, \text{ and } A = -\frac{e^2}{2} - B;$$

whence

$$g = G\left\{1 + \sin^2 l\left(\frac{5}{2}m - e + \frac{2}{7}B - \frac{17}{14}me\right) - \sin^2 l\cos^2 l\left(\frac{5}{2}me - \frac{e^2}{2} + 3A\right)\right\},$$

which is the same expression as that obtained by Mr. AIRY in the Philosophical Transactions, 1826, except that instead of e or $\frac{a-c}{a}$, the symbol is used to represent $\frac{a-c}{c}$.

24. The circumstance of the terms arising from the differentiation of V with respect to a vanishing, affords an easy method of extending CLAIRAUT's theorem indefinitely, without calculating the value of V.

It may be shown independently, that these terms cancel each other in all cases.

The $(n+1)$th term of V for the portion including the point is

$$2\pi\varrho r^n\int_{-1}^{1} d\mu' P_n\left\{\int_r^R \frac{\Phi r'}{r'^{n-1}}dr' + \mu'^2\int_r^R \frac{F r'}{r'^{n-1}}dr' + \mu'^4\int_r^R \frac{\Pi r'}{r'^{n-1}}dr' + \ldots\right\}.$$

The corresponding term for the other portion is

$$\frac{2\pi\varrho}{r^{n+1}}\int_{-1}^{1} d\mu' P_n\left\{\int_0^r r'^{n+2}\Phi r'dr' + \mu'^2\int_0^r r'^{n+2}F r'dr' + \mu'^4\int_0^r \Pi r'.r'^{n+2}dr' + \ldots\right\};$$

and it is evident, by inspection of these functions, that that portion of the sum of their partial differential coefficients, which arises from differentiating with respect to r under the signs of integration, is equal to zero. This being the case, it is not necessary to know the forms of the functions $\mathrm{M}\,a$, $\mathrm{N}\,a$, $\mathrm{P}\,a$, &c., nor their numerical coefficients; but only the functions of μ, by which they are respectively multiplied; and these are LAPLACE's coefficients.

Thus the equation of equilibrium at the surface would be

$$\mathrm{C} = \frac{\mathrm{M}\,a}{3\,r} - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{\mathrm{N}\,a}{r^3} + \frac{1}{9}\left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right)\frac{\mathrm{P}\,a}{r^5} - \left(\mu^6 - \frac{15}{11}\mu^4 + \frac{5}{11}\mu^2 - \frac{5}{231}\right)\frac{\mathrm{Q}\,a}{r^7}$$

$$+ \frac{1}{2}(1 - \mu^2)\,r^2\,m\,\mathrm{G}, \text{ where } \frac{1}{r} = \frac{1}{a}(1 + \varepsilon\,\mu^2 + \mathrm{A}\,\mu^4 + \mathrm{D}\,\mu^6), \text{ suppose.}$$

Expand r, recollecting that $\mathrm{N}\,a$, $\mathrm{P}\,a$, and $\mathrm{Q}\,a$ are of the 1st, 2nd, and 3rd orders respectively, and we have three equations to determine ε, A and D.

By differentiating C with respect to r, and eliminating $\mathrm{N}\,a$, $\mathrm{P}\,a$, and $\mathrm{Q}\,a$ by these three equations, we have the resolved force in r, which divided by the cosine of the angle of the vertical gives g exactly as in (23.).

It is evident that this may be carried on indefinitely; and to any order, without finding g for the next lower order.

METEOROLOGICAL JOURNAL,

KEPT BY THE ASSISTANT SECRETARY,

AT THE APARTMENTS OF THE

ROYAL SOCIETY,

BY ORDER OF

THE PRESIDENT AND COUNCIL.

OBSERVANDA.

Height of the Cistern of the Barometer above the plinth at Waterloo Bridge....83 feet 2 inches.
_____ above the mean level of the sea .....97 feet.
Height of the receiver of the Rain Gauge above the court of Somerset House ..79 feet.
The External Thermometer is 2 feet higher than the Barometer Cistern.
The Thermometers are graduated to Fahrenheit's scale.
The Barometer is divided into inches and tenths.
The Hours of Observation are of Mean Time, the day beginning at Midnight.
The *daily* observations of the Barometer are *not* corrected.
The *monthly means* are corrected for capillarity and temperature by the Table contained in Mr. Baily's paper in *Phil. Trans.* for 1837.

METEOROLOGICAL JOURNAL FOR JULY AND AUGUST, 1840.

1840.		9 o'clock, A.M.			3 o'clock, P.M.			Dew Point at 9 A.M. Aug. Psych.	Diff. of Wet and Dry Thermometer.	External Thermometers.				Rain in Inches and 100ths.	Direction of the Wind at 9 A.M.	REMARKS.
		Barometer uncorrected.		Att. Ther.	Barometer uncorrected.		Att. Ther.			Fahrenheit.		Self-registering				
		Flint Glass.	Crown Glass.		Flint Glass.	Crown Glass.				9 A.M.	3 P.M.	Lowest	Highest			
JULY	W 1	29.922	29.916	65.5	29.846	29.840	66.0	58	07.1	62.0	65.2	52.4	68.2	.133	S	Cloudy—light wind throughout the day. Evening, Overcast.
	T 2	29.774	29.768	63.7	29.728	29.722	65.9	58	05.8	61.2	63.7	59.6	66.6		SE	A.M. Overcast—light rain—brisk wind, as also high wind throughout the night. P.M. Overcast—light wind. Ev. Slight rain.
	F 3	29.540	29.532	64.4	29.534	29.530	65.9	58	05.2	58.8	62.9	56.5	66.5	.033	W	A.M. Overcast—light rain and wind. P.M. Fine—light clouds and wind. Evening, Overcast—light wind.
	S 4	29.878	29.874	81.5	29.902	29.896	67.5	54	09.0	61.3	67.4	53.7	69.2	.080	W	A.M. Fine—li. clds. & wind. P.M. Cldy.—brisk wind. Ev. The like.
	⊙ 5	29.732	29.728	72.0	29.706	29.700	66.4	56	08.7	63.3	63.4	58.7	68.6		W	A.M. Cloudy—brisk wind throughout the night. P.M. Overcast—brisk wind, with showers. Evening, Fine and starlight.
	M 6	29.810	29.804	68.0	29.686	29.678	66.9	55	07.5	62.6	63.8	53.3	75.3	.016	S var.	Cloudy—brisk wind throughout the day. Ev. Light rain—high wind.
	T 7	29.634	29.630	77.2	29.706	29.698	65.8	55	07.5	59.7	64.6	50.6	71.7	.138	W	A.M. Dark heavy clouds—brisk wind, with light showers. P.M. Fine—light clouds—brisk wind. Evening, Overcast.
	W 8	29.822	29.816	64.8	29.780	29.772	64.7	55	07.1	61.2	62.7	53.8	66.3	.025	SW	A.M. Cloudy—light wind. P.M. Light rain and wind. Ev. Overcast—heavy rain.
	T 9	29.878	29.872	69.0	29.956	29.948	64.9	53	07.1	59.4	65.7	51.3	67.3	.500	NW	A.M. Overcast—light wind and showers. P.M. Cloudy—light wind. Evening, Fine and starlight.
	F 10	29.996	29.988	64.2	29.960	29.952	63.9	53	05.9	60.3	62.0	51.3	67.7		W	A.M. Cloudy—light wind. P.M. Fine—light clouds. Ev. Cloudy.
	S 11	30.002	29.998	75.0	29.976	29.970	65.2	54	08.3	60.2	62.3	52.6	66.8		NW	Fine—light clouds and wind throughout the day. Ev. The like.
	⊙ 12	30.012	30.004	72.4	29.982	29.974	64.8	53	08.0	58.3	61.7	52.2	69.2		NW	A.M. Fine—li. clouds & wind. P.M. Cloudy—li. wind. Ev. The like.
	M 13	30.012	30.004	68.3	30.088	30.080	62.9	53	05.9	55.7	59.5	49.4	62.7		NW	A.M. Cloudy—light wind. P.M. Light rain and wind. Evening, Fine and moonlight.
	○ T 14	30.316	30.310	71.2	30.294	30.286	63.7	50	07.1	59.4	68.7	50.2	70.5		SW	Fine—light clouds and wind throughout the day. Evening, Fine and moonlight.
	W 15	30.304	30.300	79.9	30.220	30.212	65.8	54	09.0	63.7	72.3	57.0	77.2		S	Fine—nearly cloudless, with light breeze throughout the day. Evening, Fine and moonlight.
	T 16	30.152	30.148	70.7	30.026	30.018	67.2	56	09.3	66.5	72.4	56.0	74.3		S	Cloudy—light breeze throughout the day. Evening, The like.
	F 17	29.998	29.992	75.7	29.926	29.918	67.3	55	08.3	62.7	67.7	52.8	74.6		SW var.	Fine—light clouds and breeze throughout the day. Ev. Overcast.
	S 18	29.778	29.772	64.4	29.784	29.776	66.2	60	05.1	63.7	65.7	58.0	69.7		W	A.M. Lightly overcast—light breeze, with very fine rain. P.M. Overcast. Evening, The like. [wind.
	⊙ 19	29.700	29.694	66.3	29.622	29.618	68.2	60	06.1	63.7	69.0	58.7	68.6		S	Cloudy—light wind throughout the day. Ev. Overcast—li. rain—high
	M 20	29.566	29.560	68.3	29.548	29.544	67.3	58	07.1	63.3	65.8	53.2	70.0	.091	S var.	A.M. Cloudy—light wind, with occasional showers. P.M. Cloudy—brisk wind. Evening, Fine and moonlight.
	T 21	29.630	29.626	76.2	29.660	29.652	67.8	59	07.6	63.7	61.7	55.3	76.0	.066	W var.	A.M. Fine—light clouds—brisk wind. P.M. Overcast—light rain. Evening, Fine and starlight.
	W 22	29.820	29.816	66.2	29.858	29.850	66.6	56	05.4	61.4	65.8	52.7	70.2	.305	W	Cloudy—light wind throughout the day. Evening, Fine & starlight.
	T 23	30.032	30.024	62.8	30.056	30.048	64.0	52	06.2	59.7	60.2	53.5	72.4		W	A.M. Cloudy—very slight rain early. P.M. Cloudy—light wind. Evening, Overcast.
	F 24	30.066	30.058	65.0	30.014	30.006	64.5	54	07.6	61.7	64.7	55.0	64.5		S	Cloudy—light wind throughout the day. Ev. Overcast—brisk wind.
	S 25	29.814	29.806	62.6	29.772	29.764	66.0	58	05.0	60.3	66.0	58.5	67.4	.016	S	A.M. Overcast—high wind—rain early—high wind throughout the night. P.M. Slight rain. Evening, The like.
	⊙ 26	29.694	29.688	63.0	29.696	29.690	64.0	56	04.4	58.8	60.0	56.3	70.2	.291	SE	A.M. Lightly overcast—heavy rain during the night. P.M. Overcast—light rain. Evening, Overcast—light showers.
	M 27	29.956	29.950	64.0	29.958	29.950	65.0	58	06.9	61.0	70.2	55.3	62.4	.105	NW	A.M. Overcast—light wind. P.M. Fine—light clouds and wind. Evening, Fine and starlight.
	● T 28	30.084	30.076	63.2	30.106	30.098	65.7	57	05.0	61.7	69.7	55.6	71.4		S	A.M. Overcast—li. wind. P.M. Fine—li. clouds & wind. Ev. Cloudy.
	W 29	30.260	30.256	78.3	30.234	30.228	67.7	57	08.4	64.7	69.6	60.2	71.8		NW	A.M. Fine—nearly cloudless—li. breeze, P.M. Cloudy. Ev. Overcast.
	T 30	30.148	30.140	67.2	30.098	30.090	68.6	61	05.6	65.4	70.9	60.2	72.0		SSW	Cloudy—light wind throughout the day. Ev. Overcast—slight rain.
	F 31	34.210	30.204	72.5	30.212	30.204	69.4	60	07.3	62.3	66.7	58.7	73.2	.016	NW	Fine—light clouds & wind throughout the day. Ev. Fine & starlight.
	MEAN.	29.921	29.915	69.1	29.901	29.894	66.0	56	06.9	61.5	65.5	54.9	69.8	Sum. 1.815	Mean Barometer corrected	9 A.M. 3 P.M. A 29.817 .. 29.805 C 29.810 .. 29.797
AUGUST	S 1	30.278	30.272	73.9	30.216	30.210	68.0	56	07.2	63.7	71.5	53.2	72.6		NW	Fine—light clouds & wind throughout the day. Ev. Fine & starlight.
	⊙ 2	30.254	30.250	76.6	30.200	30.192	69.3	60	08.0	66.7	78.0	55.3	79.0		S	Fine—li. clouds & breeze throughout the day. Ev. Fine & starlight.
	M 3	30.244	30.240	82.9	30.180	30.172	72.3	62	07.6	70.0	79.3	61.0	79.3		S	Fine—nearly cloudless throughout the day. Ev. Fine and starlight.
	T 4	30.188	30.184	76.5	30.132	30.126	72.0	58	07.0	67.5	77.2	61.0	80.0		NW	A.M. Light fog. P.M. Fine—light clouds. Evening, The like.
	W 5	30.150	30.196	71.0	30.098	30.094	71.0	61	05.5	67.2	71.7	59.0	68.5		SE	Fine—light clouds throughout the day. Evening, The like.
	T 6	30.034	30.030	74.6	29.992	29.986	72.0	60	06.8	69.6	76.6	60.0	73.0		ENE	Fine—light clouds throughout the day. Evening, The like.
	F 7	30.020	30.016	74.0	30.024	30.020	73.0	61	09.7	71.3	72.0	64.0	78.5		NE	A.M. Fine—light clouds and breeze. P.M. Lightly overcast—brisk wind. Evening, The like.
	S 8	30.150	30.144	69.0	30.148	30.142	71.0	59	05.0	65.4	73.0	59.0	76.0		NNE	Fine—li. clouds—brisk wind throughout the day. Ev. The like.
	⊙ 9	30.292	30.286	70.0	30.322	30.318	69.8	54	04.4	60.2	75.5	52.4	75.5		N	A.M. Lightly overcast—light wind. P.M. Fine—light clouds and wind. Evening, The like.
	M 10	30.098	30.094	74.0	29.984	29.978	72.0	58	07.3	69.5	76.5	60.0	77.0		SW	A.M. Overcast—li. wind. P.M. Fine—li. clouds. Ev. Lightly overcast.
	T 11	29.614	29.610	68.0	29.574	29.570	68.5	60	03.0	63.5	69.0	62.0	79.7	.058	S	A.M. Owt.—li. wind. P.M. Fine—li. clouds. Ev. Lightly overcast.
	W 12	29.654	29.650	71.0	29.636	29.632	70.0	57	08.1	65.5	67.3	57.0	70.5	.069	W	A.M. Fine—light clouds and wind. P.M. Lightly overcast. Evening, Overcast—heavy rain.
	○ T 13	29.656	29.652	70.0	29.621	30.620	68.5	54	07.0	63.0	67.6	55.0	70.0	.027	SW	A.M. Overcast—light wind. P.M. Fine—light clouds—brisk wind. Evening, The like.
	F 14	29.530	29.526	66.0	29.604	29.600	67.2	54	05.0	61.0	61.0	56.0	70.0	.058	W	A.M. Overcast. P.M. Overcast—heavy showers. Evening, The like.
	S 15	29.818	29.814	65.0	29.832	29.828	66.0	54	05.5	61.2	65.8	53.5	67.0	.213	SW var.	A.M. Heavy clouds—brisk wind. P.M. Fine—light clouds, with occasional showers. Evening, Overcast.
	⊙ 16	29.968	29.962	66.0	29.904	29.900	66.0	55	06.5	63.0	68.2	54.5	67.5	.072	W	Fine—light clouds nearly the whole of the day—heavy shower at 3h. P.M. Evening, Fine—light clouds.
	M 17	29.206	29.202	63.0	29.264	29.260	64.5	59	02.0	59.8	59.5	58.0	72.0	.159	SE var.	A.M. Overcast—light rain—high wind. P.M. Fine—light clouds—brisk wind. Rainbow 5 p. m. Evening, Overcast.
	T 18	29.374	29.370	61.5	29.552	29.548	62.0	54	05.0	59.5	61.8	52.5	63.2	.175	W	Overcast—brisk wind throughout the day, also the evening.
	W 19	29.712	29.708	60.0	29.766	29.762	63.0	57	00.0	57.0	67.5	57.0	63.8	.369	SE	Overcast—light rain & wind nearly throughout the day. Evening, Overcast.
	T 20	30.016	30.012	63.5	30.002	29.998	66.0	56	05.5	66.0	72.8	60.0	69.0	.061	S	Lightly overcast throughout the day. Evening, The like.
	F 21	29.954	29.950	69.2	29.894	29.888	69.0	58	06.4	69.0	75.5	60.0	70.5		SE	A.M. Lightly overcast—light fog. P.M. Fine—light clouds and breeze. Evening, Fine—light clouds.
	S 22	29.836	29.832	67.2	29.824	29.818	69.0	62	02.2	63.5	71.6	62.5	77.8		SW	A.M. Overcast—light wind. P.M. Fine—light clouds and wind. Evening, Lightly overcast.
	⊙ 23	29.998	29.994	72.0	29.982	29.978	69.0	59	05.5	63.0	70.5	56.0	72.5		WSW	A.M. Heavy clouds—light wind. P.M. Fine—light clouds and wind. Evening, Dark heavy clouds—brisk wind.
	M 24	30.050	30.044	70.0	29.996	29.990	68.6	57	04.0	62.5	70.2	56.0	72.5		SW	A.M. Lightly overcast—li. wind. P.M. Fine—li. clouds. Ev. The like.
	T 25	30.038	30.034	72.0	30.000	29.994	68.2	55	06.5	63.5	70.0	54.4	73.5		WSW	A.M. Fine—light clouds. P.M. Lightly overcast. Ev. Overcast.
	W 26	30.060	30.054	66.0	30.070	30.066	67.3	61	03.5	65.0	68.5	61.0	73.0		NW	Overcast—light wind throughout the day. Evening, The like.
	● T 27	30.110	30.104	65.0	30.052	30.048	67.2	59	03.0	61.5	69.5	56.7	70.8		W	Overcast throughout the day, as also the evening.
	F 28	30.060	30.054	65.8	30.080	30.076	67.0	61	03.0	63.5	69.6	60.0	72.0		W	A.M. Lightly overcast. P.M. Fine—li. clouds. Evening, Lightly overcast.
	S 29	30.204	30.198	66.0	30.200	30.194	67.0	60	01.1	60.9	67.2	60.0	72.0	.011	E	A.M. Overcast—light fog and haze. P.M. Lightly overcast. Evening, Light fog.
	⊙ 30	30.128	30.124	68.5	30.086	30.082	69.0	62	04.5	66.8	77.5	58.0	69.0		E	A.M. Overcast—light fog. P.M. Fine—light clouds. Ev. Overcast.
	M 31	30.184	30.180	66.8	30.124	30.120	68.7	59	04.0	63.0	72.0	60.0	79.5		NE	A.M. Overcast—light wind. P.M. Fine—light clouds—brisk wind. Evening, Cloudy.
	MEAN.	29.964	29.961	69.2	29.947	29.942	68.5	58	05.2	64.3	70.8	57.9	72.7	Sum. 1.272	Mean Barometer corrected.....	9 A.M. 3 P.M. A 29.899 .. 29.944 C 29.885 .. 29.838

METEOROLOGICAL JOURNAL FOR SEPTEMBER AND OCTOBER, 1840.

1840.		9 o'clock, A.M.			3 o'clock, P.M.			Dew Point at 9 A.M. Sat. Fair.	Diff. of Wet and Dry Bulb Thermometer.	External Thermometers.				Rain in inches and 100ths.	Direction of the Wind at 9 A.M.	REMARKS.
		Barometer uncorrected.		Att. Ther.	Barometer uncorrected.		Att. Ther.			Fahrenheit.		Self-registering				
		Flint Glass.	Crown Glass.		Flint Glass.	Crown Glass.				9 A.M.	3 P.M.	Lowest	Highest			
	T 1	30.034	30.026	66.7	29.904	29.896	69.7	63	04.0	65.9	74.0	54.3	72.9		N	(A.M. Cloudy—light wind. P.M. Fine—nearly cloudless—light wind. Evening, Fine and starlight.
	W 2	29.756	29.748	77.9	29.674	29.668	71.6	66	06.8	69.8	70.7	63.9	76.2		S	(A.M. Fine—nearly cloudless. P.M. Fine—light clouds and wind. Evening, Overcast—heavy rain.
	T 3	29.664	29.656	66.8	29.748	29.740	68.2	61	06.0	60.3	62.7	56.3	75.7	.236	W	(A.M. Overcast—li. wind—rain early. P.M. Fine—li. clouds and wind. Ev. Fine & starlight. (Fine and starlight.
	F 4	29.826	29.818	69.7	29.826	29.818	65.3	58	07.2	61.8	60.4	52.6	64.2		S	A.M. Cloudy—light wind. P.M. Cloudy—light rain and wind. Ev.
	S 5	30.012	30.004	66.3	30.042	30.034	65.3	57	05.7	60.0	65.2	51.2	64.8	.122	W	(A.M. Light haze and wind. P.M. Cloudy—light wind. Evening, Fine and starlight.
	⊙ 6	30.276	30.268	69.0	30.228	30.220	64.7	55	05.6	59.8	68.3	50.8	67.2		S	
	M 7	30.132	30.124	68.5	30.088	30.080	65.7	58	06.3	62.6	65.8	57.3	69.8		SW	A.M. Fine—nearly cloudless. P.M. Fine—light clouds. Ev. Cloudy. (A.M. Fine—light clouds and wind. P.M. Cloudy—light rain and wind. Evening, Cloudy.
	T 8	30.180	30.172	68.0	30.158	30.150	65.0	56	04.5	58.4	64.7	52.3	68.0		NW	(A.M. Fine—nearly cloudless—light fog. P.M. Thick haze. Evening, Cloudy—very slight rain.
	W 9	30.044	30.036	62.8	29.996	29.988	63.8	57	05.2	62.3	65.8	56.7	66.7		S	
	T 10	30.014	30.006	64.0	30.020	30.012	65.2	59	05.2	61.3	65.7	60.7	68.0	.041	WNW	Overcast—brisk wind throughout the day, as also the evening.
○	F 11	29.994	29.986	61.9	29.952	29.944	63.7	55	03.8	60.0	62.8	51.9	66.8		SW	(A. M. Overcast—rain early—high wind throughout the night. P.M. Fine—light clouds. Evening, Fine and starlight.
	S 12	30.024	30.016	66.4	29.966	29.958	61.5	52	05.9	56.8	60.8	47.2	65.3		W	(Fine—light clouds and wind and brisk throughout the day. Ev. Fining, Fine and starlight.
	⊙ 13	29.900	29.894	64.9	29.764	29.758	60.0	48	05.9	53.8	58.6	45.2	62.2		SW	Fine—li. clouds and wind throughout the day. Ev. Fine and clear.
	M 14	29.474	29.466	55.9	29.308	29.300	57.9	49	06.1	54.3	53.2	46.0	60.7		S	A.M. Light Fog. P.M. Fine—light clouds. Evening, The like.
	T 15	29.314	29.306	67.3	29.306	29.300	57.4	50	04.0	51.2	54.3	45.9	59.4	.283	W	(A.M. Overcast—light wind. P.M. Overcast—light rain and wind—great depression in barometer. Evening, Heavy rain.
	W 16	28.892	28.888	62.4	28.858	28.852	57.7	49	06.8	55.7	53.8	46.9	60.8	.130	S	(Fine—light clouds and wind throughout the day. Evening, Overcast—light rain—high wind.
	T 17	29.550	29.542	60.7	29.660	29.652	56.6	48	06.2	55.3	58.2	45.7	60.0	.111	SW var.	A.M. Fine—li. clouds & wind. P.M. Overcast—li. showers—brisk wind. Ev. Fine and starlight. (and starlight.
	F 18	29.896	29.890	55.3	29.888	29.880	55.0	45	05.3	51.3	57.7	41.7	59.3		W	Fine—light clouds—brisk wind throughout the day. Evening, Fine (A.M. Fine—light clouds and fog. P.M. Fine—light clouds & wind
	S 19	29.880	29.872	56.3	29.908	29.900	56.2	49	04.6	53.3	54.6	49.3	59.7		NNW	and mist & rain-bow, with slight rain. Evening, Fine and clear.
	⊙ 20	30.036	30.028	52.3	30.036	30.030	54.0	44	05.9	51.3	55.0	42.4	56.6		WNW	(A.M. Fine—light clouds—high wind. P.M. Cloudy—brisk wind. Ev. Fine and starlight. (Fine and starlight.
	M 21	30.026	30.018	55.0	29.946	29.938	56.0	45	05.3	54.5	58.8	45.0	57.8		S	A.M. Fine—nearly cloudless—li. fog. P.M. Overcast—li. wind. Ev.
	T 22	29.500	29.494	55.4	29.500	29.496	55.3	50	04.8	55.2	49.7	52.4	62.5	.083	S	(A.M. Fine—light clouds and wind. P.M. Cloudy—light brisk wind. Evening, Cloudy—light rain.
	W 23	29.402	29.396	52.3	29.498	29.492	54.7	45	04.8	51.4	57.0	44.6	56.7	.333	WSW	Overcast—li. rain—high wind nearly the whole day. Ev. Cloudy.
	T 24	29.530	29.522	53.0	29.574	29.566	54.9	48	04.8	53.2	56.3	48.0	58.8		E	A.M. Fine—light clouds—brisk wind. P.M. Fine—dark clouds—brisk wind. Evening, Fine and starlight.
●	F 25	29.796	29.788	53.3	29.944	29.936	54.8	49	02.3	50.2	54.3	50.0	59.3	.097	NW	(A.M. Fine—light clouds and wind. P.M. Cloudy, with occasional light showers. Evening, Overcast—light rain.
	S 26	30.062	30.054	53.4	29.966	29.958	55.0	46	03.0	50.7	57.7	45.4	56.5	.019	SW	A.M. Overcast—li. rain & wind. P.M. Cloudy—brisk wind. Ev. Li. fog.
	⊙ 27	30.014	30.006	55.7	29.980	29.976	57.0	48	04.5	55.4	61.8	49.0	60.6	.130	S	(A.M. Fine—light clouds and wind. P.M. Cloudy—light rain and wind. Evening, Overcast—light rain and wind.
	M 28	29.790	29.782	55.4	29.654	29.648	57.2	52	04.4	56.3	58.0	51.3	63.8		SE	A.M. Fine—li. clouds. P.M. Lightly overcast. Ev. Fine & starlight. (Overcast—high wind, with light rain nearly the whole of the day, as also the evening.
	T 29	29.624	20.618	58.2	29.630	29.624	58.5	53	04.4	55.5	56.3	50.2	60.3	.269	S	(A.M. Fine—nearly cloudless—brisk wind. P.M. Fine—light clouds. Evening, Fine and starlight.
	W 30	29.998	29.990	55.5	30.004	29.996	57.2	49	04.7	52.8	56.4	46.3	59.0		W	(A.M. Fine—light clouds and wind. P.M. Cloudy—light wind. Evening, Dark heavy clouds.
	MEAN.	29.821	29.814	61.0	29.801	29.794	60.2	52	05.1	57.3	60.0	50.0	63.3	Sum. 1.854		Mean Barometer corrected ..... { 9 A.M. 29.738 .. 3 P.M. 29.780
	T 1	30.022	30.016	55.2	30.044	30.036	56.2	50	04.6	55.3	57.3	51.7	57.7		S	Overcast—light fog and wind throughout the day, as also the evening.
	F 2	30.172	30.164	55.3	30.172	30.164	57.6	52	04.1	53.7	57.7	51.3	58.7		W	(A.M. Cloudy—light wind. P.M. Fine—light clouds and wind. Ev. Fine and starlight.
	S 3	30.194	30.186	53.8	30.148	30.140	54.9	46	03.3	47.7	52.5	44.3	59.0		NW	(A.M. Fine—li. clouds & wind. P.M. Cloudy—li. wind. Ev. The like.
	⊙ 4	30.084	30.076	53.3	30.080	30.074	55.5	49	02.8	51.2	55.0	46.3	54.6		NNW	(A.M. Lightly overcast—light rain—brisk wind. P.M. Hazy clouds —light wind. Evening, Fine and starlight.
	M 5	30.100	30.092	52.2	30.078	30.070	53.4	48	03.0	49.5	49.4	46.4	57.2		NW	A.M. Ovet.—li. wind. P.M. Cloudy—li. showers. Ev. Fine & starlight.
	T 6	30.126	30.118	51.0	30.118	30.110	53.2	45	02.6	45.4	52.4	42.6	55.3	.116	NW	(Cloud—light fog and wind throughout the day. Evening, Fine and starlight—frosty.
	W 7	30.114	30.106	49.5	30.116	30.108	51.2	42	02.6	43.7	52.8	39.0	53.6		SSW	A.M. Fine—light clouds, with light fog and wind. P.M. Cloudy—light wind. Evening, Fine and starlight—light fog.
	T 8	30.266	30.258	48.0	30.252	30.244	48.7	41	02.4	40.7	50.3	36.8	53.9		S	A.M. Fine & cloudless—li. fog. P.M. Foggy. Ev. Fine & moonlight.
	F 9	30.326	30.318	46.7	30.282	30.276	48.0	39	02.4	40.3	53.8	40.0	51.6		WSW	(A.M. Thick fog—light wind. P.M. Fine—nearly cloudless. Ev. Fine and moonlight—light fog.
	S 10	30.318	30.310	46.7	30.292	30.284	48.8	41	03.4	45.7	55.2	39.8	55.4		N	A.M. Thick fog—li. wind. P.M. Fine—nearly cloudless. Ev. Fine and starlight—light fog. (Fine & moonlight.
○	11	30.400	30.392	49.3	30.408	30.402	51.8	46	03.4	52.5	57.5	44.4	56.7		NE	A.M. Lightly overcast—light fog and wind. P.M. Fine—li. wind. Ev.
	M 12	30.604	30.596	49.6	30.592	30.584	51.6	46	02.9	50.3	55.7	44.7	59.3		N	(Fine—nearly cloudless—light wind throughout the day. Evening, Fine and moonlight—light fog.
	T 13	30.590	30.582	48.8	30.530	30.522	50.7	43	02.1	43.2	55.2	42.2	57.3		NNW	(A.M. Thick fog—light wind. P.M. Fine—light clouds and wind. Evening, Fine and moonlight—light fog.
	W 14	30.404	30.398	48.6	30.308	30.300	50.4	42	01.2	43.2	58.8	39.7	56.6		SW	(Fine—nearly cloudless—light fog throughout the day. Evening, Fine and moonlight—light fog.
	T 15	30.258	30.250	50.4	30.152	30.144	51.3	42	02.3	43.7	53.8	41.9	60.7		S	A.M. Fine—light fog. P.M. Hazy—light fog. Ev. Cloudy—light fog.
	F 16	29.936	29.928	50.3	29.830	29.822	51.8	46	03.5	48.3	55.3	42.0	54.8		S	(A.M. Light clouds and wind. P.M. Overcast—light wind. Evening, Overcast—light rain.
	S 17	29.812	29.804	53.0	29.900	29.892	53.8	49	02.5	54.8	53.3	48.0	56.7	.066	NW	(Lightly overcast—light wind throughout the day. Evening, Overcast—deposition.
	⊙ 18	30.026	30.018	52.2	29.936	29.932	53.0	47	02.5	50.8	53.5	49.0	56.8		SSW	Overcast throughout the day. Ev. Overcast—light rain—brisk wind.
	M 19	29.704	29.698	53.3	29.726	29.718	54.3	48	05.2	53.5	54.8	49.0	55.7	.133	NW	(A.M. Cloudy—brisk wind. P.M. Fine—light clouds—brisk wind. Evening, Fine and starlight—high wind.
	T 20	30.100	30.094	51.3	30.104	30.096	52.2	45	04.8	47.7	52.8	45.8	56.6		NW	A.M. Fine and cloudless—brisk wind. P.M. Fine—light clouds and wind. Evening, Fine and starlight.
	W 21	30.156	30.150	49.7	30.120	30.112	51.0	41	04.0	45.7	51.8	42.2	54.6		NW	Fine—light clouds and wind throughout the day. Ev. Overcast.
	T 22	29.904	29.898	50.0	29.856	29.848	50.8	45	02.1	48.8	51.7	44.8	52.8		S	(Overcast—slight rain—light wind throughout the day. Evening, Fine and starlight.
	F 23	29.960	29.952	49.6	29.856	29.848	52.0	42	02.4	45.2	52.2	43.7	52.7	.036	SW	A.M. Fine—li. clouds & wind. P.M. Overcast—light rain and wind.
	S 24	29.556	29.548	49.6	29.556	29.548	50.2	43	03.2	45.2	47.2	42.5	53.2	.286	W	(A.M. Fine—light clouds—brisk wind. P.M. Fine—light clouds—at 9 P.M. heavy shower of hail. Evening, Fine and starlight.
●	25	29.784	29.776	47.0	29.832	29.826	47.7	38	03.2	42.2	48.2	39.3	50.6	.102	NW	(Fine—nearly cloudless—light wind throughout the day. Evening, Fine and starlight. (cast—li. rain.
	M 26	29.886	29.878	44.3	29.826	29.818	46.2	37	02.0	39.2	47.8	35.4	49.3		W	A.M. Overcast—li. fog and wind. P.M. Lightly overcast. Ev. Over-
	T 27	29.178	29.172	48.7	29.152	29.146	49.5	45	01.9	51.3	49.3	38.6	52.8	.302	S	(A.M. Overcast—light rain and wind, as also throughout the night.
	W 28	29.136	29.130	46.8	29.210	29.202	49.0	42	02.0	42.7	47.9	41.3	52.3	.061	S	P.M. Cloudy—light wind. Evening, Fine and starlight.
	T 29	29.340	29.336	46.7	29.310	29.304	47.7	43	01.2	45.6	47.2	41.4	49.4	.033	NE	A.M. Fine—light clouds & wind—rain early. P.M. Lightly cloudy. Evening, Overcast—light rain. (Overcast—li. fog.
	F 30	29.522	29.514	47.4	29.528	29.522	49.7	44	02.3	47.3	51.7	40.3	49.6	.338	E	Overcast—very fine rain—light wind nearly the whole of the day. Ev.
	S 31	29.564	29.558	48.5	29.564	29.558	50.7	45	02.0	47.3	51.5	43.2	52.8		ENE	A.M. Fine—light clouds and wind. P.M. Fine, with occasional showers. Evening, Fine and starlight.
	MEAN.	29.985	29.978	50.0	29.932	29.924	51.4	44	02.8	47.2	52.7	43.2	54.8	Sum. 1.473		A.M. Lightly overcast—light wind. P.M. Fine—light clouds. Evening, Fine and starlight.
																Mean Barometer corrected ..... { 9 A.M. 29.931 .. 3 P.M. 29.975

METEOROLOGICAL JOURNAL FOR NOVEMBER AND DECEMBER, 1840.

1840.		9 o'clock, A.M.			3 o'clock, P.M.			Dew Point at 9 A.M.	Dew Point at 3 P.M.	External Thermometers.				Rain in inches. Reg'd at 9 A.M.	Direction of the Wind at 9 A.M.	REMARKS.
		Barometer uncorrected.			Barometer uncorrected.					Fahrenheit.		Self-registering				
		Flint Glass.	Crown Glass.	Att. Ther.	Flint Glass.	Crown Glass.	Att. Ther.			9 A.M.	3 P.M.	Lowest	Highest			
NOVEMBER	☉ 1	29.644	29.638	48.8	29.488	29.484	50.0	47	01.7	47.2	50.5	44.9	53.7		E	{Overcast—deposition—lt. wind throughout the day. Evening, Light rain—high wind.
	M 2	29.538	29.532	50.4	29.544	29.538	51.8	46	03.5	51.2	54.7	46.5	53.3	.050	SE	{A.M. Fine—lt. clouds—high wind—very high wind throughout the night. P.M. Fine—lt. clouds and wind. Ev. Fine & starlight.
	T 3	29.432	29.428	51.3	29.398	29.394	52.4	48	02.0	51.5	54.0	47.6	56.4		NE	A.M. Lightly overcast. P.M. Fine—light clouds. Ev. Overcast.
	W 4	29.288	29.282	51.8	29.292	29.286	53.2	49	01.3	50.3	53.3	49.0	55.6	.052	S	{A.M. Overcast—deposition—lt. fog and wind—rain during the night.
	T 5	29.202	29.196	51.9	29.296	29.292	53.4	49	02.1	50.3	52.5	48.8	55.8	.302	S	{P.M. Fine—light clouds. Evening. Fine and starlight.
	F 6	29.056	29.050	50.6	28.980	28.976	51.4	45	02.0	47.3	48.7	45.8	54.4	.133	SE	{A.M. Dark heavy clouds—high wind—heavy rain early. P.M. Fine —light clouds and wind. Ev. Overcast—light rain—brisk wind.
	S 7	29.138	29.132	51.4	29.168	29.162	52.3	47	02.3	48.8	50.4	48.0	51.7	.286	SSW	{A.M. Overcast—light rain—high wind. P.M. Fine—light clouds.
	⊙ 8	29.360	29.352	49.7	29.358	29.352	51.0	45	02.9	47.0	51.5	44.7	55.3	.083	S	{A.M. Fine—light clouds and wind—very high wind throughout the night, P.M. Fine—lt. clouds—occasional rain. Ev. Fine & starlight.
	M 9	29.106	29.100	49.9	29.142	29.136	51.2	46	03.5	48.8	51.7	46.0	53.8	.091	S	Fine & cloudless throughout the day. Ev. Fine & moonlight—lt. wind.
	T 10	29.074	29.070	49.3	29.166	29.158	50.2	45	01.9	44.5	48.8	44.2	53.5	.105	SSW	{A.M. Fine—nearly cloudless—brisk wind—very high wind early, with rain. P.M. Cloudy—light rain. Ev. Overcast—light rain.
	W 11	29.332	29.326	47.3	29.166	29.160	48.8	43	01.9	44.2	45.3	42.2	50.3		S	Fine—light clouds and wind throughout the day. Ev. Fine & starlight.
	T 12	29.528	29.520	46.6	29.534	29.528	47.4	40	01.7	41.3	46.8	40.6	49.8	.114	S	A.M. Lt. fog & wind—P.M. Overcast—lt. rain & wind. Ev. Overcast.
	F 13	28.828	28.822	47.8	28.606	28.600	50.2	45	02.0	48.7	51.7	41.7	51.0	.272	S	Fine—light clouds and wind throughout the day. Ev. Lightly overcast.
	S 14	29.044	29.038	48.7	29.160	29.154	49.4	44	03.1	46.2	48.4	45.0	53.4	.230	S	{A.M. Fine—lt. cldy.—hi. wind—rain, with very hi. wind throughout the night. P.M. Dk. cldy.—hail & rain—hi. wi. Ev. Overct—h. rain.
	⊙ 15	29.512	29.504	46.2	29.550	29.546	47.0	39	01.6	39.3	45.2	38.6	50.6		S	Fine—nearly cloudless throughout the day. Ev. Fine and moonlight.
	M 16	29.084	29.076	48.7	29.200	29.194	52.0	49	01.9	55.5	57.3	38.5	57.9	.083	SW var.	{A.M. Fine—nearly cloudless—light wind. P.M. Overcast. Ev. Over-cast—light rain. Ev. Lightly overcast.
	T 17	29.600	29.594	51.3	29.454	29.448	51.3	47	02.6	48.2	45.3	46.7	60.6		S var.	A.M. Overcast—light rain—very high wind. P.M. Fine—light clouds.
	W 18	29.814	29.806	49.0	29.752	29.748	47.3	43	03.1	41.2	36.3	41.4	54.6	.447	NW	{Overcast—light rain—brisk wind. P.M. Snow and rain—brisk wind. Ev. Rain and wind.
	T 19	29.800	29.796	44.4	29.852	29.846	45.6	41	01.4	39.8	43.3	37.2	42.4	.450	N	{Ev. Fine and starlight.
	F 20	30.112	30.104	42.0	30.090	30.082	43.6	33	02.9	35.7	42.8	35.0	44.7		NW	A.M. Overcast—light rain and wind. P.M. Cloudy—light wind.
	S 21	29.526	29.518	42.8	29.218	29.210	44.6	41	01.9	43.3	46.0	35.0	44.3	.033	S	Fine—light clouds and wind throughout the day. Ev. fine & starlight.
	⊙ 22	29.794	29.786	43.2	29.920	29.916	44.0	38	03.3	40.3	42.3	40.0	47.3	.211	NNW	{A.M. Fine—light clouds—high wind throughout the day—very high wind during the night. Ev. fine and starlight.
	M 23	30.070	30.062	41.2	30.000	29.994	43.0	33	02.6	38.4	45.3	35.6	43.5		S	Overcast—light wind and rain nearly the whole of the day. Ev. lt. fog.
	● T 24	30.140	30.134	46.3	30.220	30.212	47.9	43	01 6	50.0	52 0	38.2	51.2		W	{A.M. Cloudy—light fog and wind. P.M. Fine—light clouds and wind. Ev. Fine and starlight.
	W 25	30.424	30.416	45.0	30.426	30.418	46.0	40	01.6	41.0	46.0	38.3	54.3		N	{A.M. Thick fog—light wind. P.M. Cloudy—light wind. Ev. Fine and starlight.
	T 26	30.460	30.452	42.0	30.416	30.408	42.3	35	02.3	37.4	43.3	36.0	47.0		E	A.M. Light fog & wind. P.M. Fine—light clouds. Ev. Fine & starlight.
	F 27	30.430	30.424	41.0	30.412	30.404	41.4	35	01.5	34.5	43.8	34.3	44.0		N	{A.M. Thick fog—light wind—white frost. P.M. Fine—light clouds. Ev. Light fog.
	S 28	30.408	30.398	39.7	30.374	30.366	39.2	35	01.0	32.8	37.7	32.8	44.5		NE	Thick fog early the whole of the day. Ev. Dense fog—sharp frost.
	⊙ 29	30.336	30.328	37.3	30.248	30.242	38.6	32	01.1	30.8	42.2	30.3	39.3		NE	A.M. Thick fog—white frost. P.M. Overcast—light fog. Ev. Cloudy.
	M 30	30.084	30.076	40.3	30.054	30.048	42.8	35	02.2	46.7	51.8	30.3	47.7		S	{Overcast—deposition—light wind throughout the day, as also the evening.
														Sum.		
	MEAN.	29.639	29.632	46.5	29.616	29.610	47.6	42	02.2	44.1	47.6	40.8	50.7	2.942	Mean Barometer corrected	9 A.M. 3 P.M. F 29.595 .. 29.569 C. 29.587 .. 29.562

DECEMBER																
	T 1	29.960	29.952	47.5	30.000	29.992	48.9	45	01.4	51.7	53.5	45.8	52.8		SE var.	Overcast—light wind throughout the day, as also the evening.
	W 2	30.188	30.180	47.9	30.248	30.240	48.0	42	02.8	44.6	46.2	44.7	55.5		WNW	Cloudy—light wind throughout the day. Evening, Fine & starlight.
	T 3	30.338	30.330	43.0	30.036	30.028	43.3	36	01.9	36.3	40.5	35.7	47.4		NW	{A.M. Fine—nearly cloudless—lt. wind. P.M. Light fog. Evening, Fine and moonlight.
	F 4	30.480	30.472	40.3	30.436	30.428	40.6	33	01 3	32.8	38.3	32.5	42.3		SW	A.M. Light fog—white frost. P.M. Light fog. Ev. Overcast—lt. fog.
	S 5	30.466	30.458	40.9	30.306	30.298	41.6	34	02.4	39.4	42.8	32.7	41.0		SW	{Light fog and wind throughout the day—sif'lenzy. Evening, Over-cast—light fog.
	⊙ 6	30.186	30.178	41.8	30.100	30.094	42.2	35	01.4	40.8	42.2	38.6	43.5		E	{A.M. Thick fog—light wind. P.M. Overcast—lt. wind. Evening, Fine and starlight.
	M 7	29.808	29.800	40.9	29.646	29.640	41.5	35	02.0	39.2	38.3	35.2	43.7		E	Lightly overcast—lt. brisk wind throughout the day, also the evening.
	T 8	29.230	29.224	39.9	29.254	29.248	41.3	34	02.6	39.7	43.2	34.2	40.3		E	{Overcast—light brisk wind, with occasional light rain during the day. Evening, Fine and moonlight.
	○ W 9	29.738	29.730	40.4	29.854	29.848	41.0	35	01.3	36.3	41.3	36.2	44.3	.066	S	Fine—lt. clouds and wind throughout the day. Ev. Fine & moonlight.
	T 10	29.970	29.962	40.0	29.936	29.928	41.3	38	01.3	40.8	43.2	36.2	42.0		E	Light fog, with lt. deposition throughout the day. Ev. Lightly cloudy.
	F 11	29.958	29.950	40.2	29.940	29.934	39.9	36	01.9	37.6	36.0	35.2	45.0		ENE	Overcast—brisk wind throughout the day, as also the evening.
	S 12	30.124	30.116	39.5	30.176	30.168	39.8	34	02.6	38.5	38.3	36.0	39.2		N	Light fog and wind throughout the day. Evening, The like.
	⊙ 13	30.282	30.276	38.2	30.280	30.272	37.9	33	02.5	33.8	34.3	33.7	39.7		N	Overcast—brisk wind throughout the day. Ev. The like, with lt. fog.
	M 14	30.236	30.228	34.0	30.220	30.212	33.2	25	*frozen*	27.3	29.8	27.3	35.4		N	{A.M. Light clouds and wind—sharp frost. P.M. Fine—light clouds and wind. Evening, Overcast—sharp frost.
	T 15	30.276	30.268	32.3	30.242	30.234	32.3	25	*ditto*	27.0	27.3	27.3	31.3		N	A.M. Overcast—light snow. P.M. Fine—light clouds. Evening, Fine and starlight.
	W 16	30.132	30.124	29.8	29.966	29.958	30.8	25	*ditto*	28.3	30.3	24.0	28.5		NW	Overcast—light snow—brisk wind throughout the day. Ev. The like.
	T 17	29.772	29.766	32.5	29.766	29.758	32.8	28	*ditto*	31.4	24.3	27.7	34.6		NE	{A.M. Overcast—light snow—brisk wind. P.M. Fine—light clouds and wind. Evening, Overcast—sharp frost.
	F 18	29.614	29.608	29.6	29.494	29.488	31.8	25	*ditto*	28.3	35.2	21.2	31.7		N	{A.M. Overcast—brisk wind—slight thaw. P.M. Overcast—con-tinued thaw. Evening, Overcast—snow and wind.
	S 19	29.588	29.580	31.8	29.568	29.560	33.0	27	*ditto*	31.3	35.0	29.2	35.5		N	{Overcast—light wind, with frost throughout the day. Evening, Overcast—thaw.
	⊙ 20	29.840	29.832	34.0	29.950	29.944	34 9	29	02.4	35.8	36.3	29.3	36 3		N	Overcast—light brisk wind throughout the day. Evening, The like.
	M 21	30.262	30.254	35.9	30.278	30.270	36.7	30	02.2	35.6	36.2	35.0	37.2		NE	{A.M. Overcast—high wind. P.M. Fine—light clouds—high wind. Evening, Overcast—brisk wind.
	T 22	30.344	30.336	35.4	30.298	30.290	34.8	28	*frozen*	32.2	32.3	32.3	37.3		ENE	{A.M. Cloudy—high wind—sharp frost. P.M. Fine—light clouds and wind. Evening, Overcast—light fog [—in clouds.
	● W 23	30.176	30.170	31.7	30.094	30.086	33.2	25	*ditto*	27.5	33.8	26.9	32.8		NNW	A.M. Light fog & wind. P.M. Fine—lt. clouds. Ev. Fine & starlight
	T 24	30.100	30.092	32.0	30.120	30.112	33.5	28	*ditto*	32.3	35.8	27.0	34.4		NNW	{A.M. Overcast—brisk wind—sharp frost. P.M. Fine—light clouds. Evening, Fine and starlight.
	F 25	30.328	30.320	34.0	30.342	30.338	33.0	28	*ditto*	30.0	33.0	28.9	36.7		N	{Light fog and wind, with sharp frost throughout the day. Evening, Thick fog—sharp frost.
	S 26	30.562	30.554	30.7	30.574	30.568	31.0	25	*ditto*	29.0	30.2	25.0	33.4		NW	Overcast—light wind throughout the day. Ev. Lt. fog—sharp frost.
	⊙ 27	30.656	30.650	31.5	30.590	30.584	32.0	26	*ditto*	32.3	34.4	29.2	32.6		NW	{A.M. Light fog and wind—sharp frost. P.M. Overcast—thaw. Evening, Overcast.
	M 28	30.408	30.400	32.3	30.342	30.334	32.9	28	*ditto*	30.3	32.8	29.4	35.3		E	A.M. Overcast—lt. wind. P.M. Light fog & wind. Ev. The like.
	T 29	30.256	30.248	31.2	30.246	33.240	31.8	26	*ditto*	26.4	32.3	24.2	33.0		S	{Light fog and wind, with sharp frost throughout the day. Evening, Overcast—rapid thaw—rain.
	W 30	30.406	30.398	31 8	30.310	30.304	32.3	27	*ditto*	27.8	34.2	26.2	37.6		S	A.M. Light fog & wind. P.M. Fine—lt. clouds. Ev. Overcast—lt. rain.
	T 31	29.886	29.878	34.0	29.930	29.924	36.5	30	01.3	41.3	43.0	28.5	42.0	.347	W	{A.M. Overcast—light fog and rain. P.M. Fine—light clouds and wind. Evening, Overcast—light rain.
														Sum.		
	MEAN.	30.121	30.114	36.3	30.082	30.075	36.9	31	01.9	34.4	36.6	31.5	38.8	.413	Mean Barometer corrected	9 A.M. 3 P.M. F. 30.104 .. 30.084 C. 30.094 .. 30.086